COHOMOLOGICAL RIGIDITY AND THE NUMBER OF HOMEOMORPHISM TYPES FOR SMALL COVERS OVER PRISMS

XIANGYU CAO AND ZHI LÜ

Abstract. In this paper we give a method of constructing homeomorphisms between two small covers over simple convex polytopes. As a result we classify, up to homeomorphism, all small covers over a 3-dimensional prism $\mathrm{P}^3(m)$ with $m \geq 3$. We introduce two invariants from colored prisms and other two invariants from ordinary cohomology rings of small covers. These invariants can form a complete invariant system of homeomophism types of all small covers over a prism in most cases. Then we show that the cohomological rigidity holds for all small covers over a prism $\mathrm{P}^3(m)$ (i.e., cohomology rings of all small covers over a $\mathrm{P}^3(m)$ determine their homeomorphism types). In addition, we also calculate the number of homeomorphism types of all small covers over a $\mathrm{P}^3(m)$.

1. Introduction

In 1991, Davis and Januszkiewicz [DJ] introduced and studied a class of $\mathbb{Z}_2^n$-manifolds (called small covers), which belong to the topological version of toric varieties. An n-dimensional small cover M^n is a closed n-manifold with a locally standard $\mathbb{Z}_2^n$-action such that its orbit space is a simple convex n-polytope P^n. As shown in [DJ], P^n naturally admits a characteristic function λ defined on the facets of P^n (here we also call λ a $\mathbb{Z}_2^n$-coloring on P^n), so that the geometrical topology and the algebraic topology of the samll cover M^n can be completely determined by the pair (P^n, λ). In other words, the Davis-Januszkiewicz theory for small covers indicates the following two key points:

- Each small cover $\pi : M^n \longrightarrow P^n$ can be reconstructed from (P^n, λ), and this reconstruction is denoted by $M(\lambda)$. Thus, all small covers over a simple convex polytope P^n correspond to all $\mathbb{Z}_2^n$-colorings on P^n, i.e., all small covers over a simple convex polytope P^n are given by $\Gamma(P^n) = \{M(\lambda) | \lambda$ is a $\mathbb{Z}_2^n$-coloring on $P^n\}$.
- The algebraic topology of a small cover $\pi : M^n \longrightarrow P^n$, such as equivariant cohomology, mod 2 Betti numbers and ordinary cohomology etc., can be explicitly expressed in terms of (P^n, λ).

In the recent years, much further research on small covers has been carried on (see, e.g. [I], [GS], [NN], [CCL], [C], [LM], [LY], [KM], [M1]-[M3]). In some sense, the classification up to equivariant homeomorphism of small covers over a simple convex polytope has been understood very well. Actually, this can be seen from the following two kinds of viewpoints: One is that two small covers $M(\lambda_1)$ and $M(\lambda_2)$

Key words and phrases. Small covers, prism, classification.
This work is supported by grants from FDUROP (No. 080705) and NSFC (No. 10671034).

over a simple convex polytope P^n are equivariantly homeomorphic iff there is an automorphism $h \in \mathrm{Aut}(P^n)$ such that $\lambda_1 = \bar{h} \circ \lambda_2$ where $\bar{h}$ induced by h is an automorphism on all facets of P^n (see [LM]); the other is that two small covers $M(\lambda_1)$ and $M(\lambda_2)$ over a simple convex polytope P^n are equivariantly homeomorphic iff their equivariant cohomologies are isomorphic as $H^*(B\mathbb{Z}_2^n; \mathbb{Z}_2)$-algebras (see [M1]). However, in non-equivariant case, the classification up to homeomorphism of small covers over a simple convex polytope is far from understood very well except for few special polytopes (see, e.g. [GS], [KM], [M2], [M3], [LY]).

In this paper we shall introduce an approach (called the *sector method*) of constructing homeomorphisms between two small covers. The basic idea of sector method is simply stated as follows: we first cut a $\mathbb{Z}_2^n$-colored simple convex n-polytope (P^n, λ) in $\mathbb{R}^n$ into two parts P_1 and P_2 by using an $(n-1)$-dimensional hyperplane H such that the section S cut out by H is an $(n-1)$-dimensional simple convex polytope with certain property. Of course, S naturally inherits a coloring from (P^n, λ). Then by using automorphisms of S and automorphisms of $\mathbb{Z}_2^n$ we can construct new colored polytopes (P'^n, λ') from (P^n, λ) (note that generally P'^n may not be combinatorially equivalent to P^n), and further obtain new small covers $M(\lambda')$ from those new colored polytopes (P'^n, λ') by the reconstruction of small covers. Moreover, we can study how to construct the homeomorphisms between $M(\lambda)$ and $M(\lambda')$. In particular, we give the method of constructing the homeomorphisms between $M(\lambda)$ and $M(\lambda')$ (see Theorems 3.2-3.3).

As an application, up to homeomorphism we shall classify all small covers over prisms. Let $\mathrm{P}^3(m)$ denote a 3-dimensional prism that is the product of $[0, 1]$ and an m-gon where $m \geq 3$. Let $\Lambda(\mathrm{P}^3(m)) = \{\lambda \big| \lambda \text{ is a } \mathbb{Z}_2^3\text{-coloring on } \mathrm{P}^3(m)\}$, and let $\Gamma(\mathrm{P}^3(m)) = \{M(\lambda) \big| \lambda \in \Lambda(\mathrm{P}^3(m))\}$. Using the sector method, we first study rectangular sectors and find seven rectangular sectors with a good twist $\Psi(\psi, \rho, v_0)$ in Section 4 (see Definition 3.3 for $\Psi(\psi, \rho, v_0)$). We then use these seven rectangular sectors to define some operations on the coloring sequences of side-faces of colored polytopes $(\mathrm{P}^3(m), \lambda)$ in Section 5. Furthermore, we show that all $(\mathrm{P}^3(m), \lambda)$ can be reduced to some canonical forms without changing the homeomorphism type of the small covers $M(\lambda)$ (see Propositions 5.2-5.4 and 5.9). In particular, we introduce two combinatorial invariants m_λ and n_λ in Section 5, and then show in Section 6 that (m_λ, n_λ) is actually a complete homeomorphism invariant of a class of small covers $M(\lambda)$ (see Corollary 6.6). In addition, we also introduce two algebraic invariants $\Delta(\lambda)$ and $\mathcal{B}(\lambda)$ in $H^*(M(\lambda); \mathbb{Z}_2)$, which can become a complete homeomorphism invariant in most cases. Both (m_λ, n_λ) and $(\Delta(\lambda), \mathcal{B}(\lambda))$ are of interest because of the nature of colored polytopes. With the help of these invariants, we can obtain the following cohomological rigidity theorem.

Theorem 1.1 (Cohomological rigidity)**.** *Two small covers $M(\lambda_1)$ and $M(\lambda_2)$ in $\Gamma(\mathrm{P}^3(m))$ are homeomorphic if and only if their cohomologies $H^*(M(\lambda_1); \mathbb{Z}_2)$ and $H^*(M(\lambda_2); \mathbb{Z}_2)$ are isomorphic as rings.*

Remark 1.1. Kamishima and Masuda showed in [KM] that the cohomological rigidity holds for small covers over a cube. In addition, Masuda in [M2] also gave a cohomological non-rigidity example of dimension 25. A further question is what extent the cohomological rigidity can extend to small covers.

In addition, we also determine the number of homeomorphism classes of all small covers in $\Gamma(\mathrm{P}^3(m))$.

Theorem 1.2 (Number of homeomorphism classes)**.** *Let $N(m)$ denote the number of homeomorphism classes of all small covers in $\Gamma(\mathrm{P}^3(m))$. Then*

$$N(m) = \begin{cases} 2 & \textit{if } m = 3 \\ 4 & \textit{if } m = 4 \\ \sum_{0 \leq k \leq \frac{m}{2}} ([\frac{k}{2}] + 1) + 6 & \textit{if } m > 4 \textit{ is even} \\ \sum_{1 \leq k \leq \frac{m}{2}} ([\frac{k}{2}] + 1) + 4 & \textit{if } m > 4 \textit{ is odd.} \end{cases}$$

The arrangement of this paper is as follows. In Section 2 we review the basic theory about small covers. In Section 3 we introduce the sector method. In Section 4 we apply the sector method to colored prisms and discuss the rectangular sectors. In Section 5 we use the rectangular sectors to define some operations on the coloring sequences of colored prisms and then determine the canonical forms of all colored prisms. In Section 6 we introduce two algebraic invariants of cohomology and then calculate such two invariants of all small covers. Finally, we complete the proofs of Theorems 1.1 and 1.2 in Section 7.

2. Theory of small covers

The purpose of this section is to briefly review the theory of small covers. Throughout the following assume that $\pi : M^n \longrightarrow P^n$ is a small cover over a simple convex n-polytope P^n. Note that a simple convex n-polytope P^n means that exactly n faces of codimension-one (i.e., facets) meet at each of its vertices. Let $\mathcal{F}(P^n) = \{F_1, ..., F_\ell\}$ denote the set of all facets of P^n.

2.1. **Coloring and Reconstruction.** Take a k-face F^k of P^n, an easy observation shows (see also [DJ, Lemma 1.3]) that $\pi^{-1}(F^k) \longrightarrow F^k$ is still a k-dimensional small cover. In particular, for any $x \in \pi^{-1}(\mathrm{int}F^k)$, its isotropy subgroup G_x is independent of the choice of x, denoted by G_F. G_F is isomorphic to $\mathbb{Z}_2^{n-k}$, and G_F fixes $\pi^{-1}(F^k)$ in M^n. In the case $k = n-1$, F^{n-1} is a facet and G_F has rank 1, so that G_F uniquely corresponds to a nonzero vector v_F in $\mathbb{Z}_2^n$. Then there is a natural map (called *characteristic function*)

$$\lambda : \mathcal{F}(P) \longrightarrow \mathbb{Z}_2^n$$

by mapping each facet F to its corresponding nonzero vector v_F in $\mathbb{Z}_2^n$ with the property $(\star)$: whenever the intersection of some facets $F_{i_1}, ..., F_{i_r}$ in $\mathcal{F}(P^n)$ is nonempty, $\lambda(F_{i_1}), ..., \lambda(F_{i_r})$ are linearly independent in $\mathbb{Z}_2^n$. Note that if each nonzero vector of $\mathbb{Z}_2^n$ is regarded as being a color, then the characteristic function λ means that each facet is colored by a color. Thus, we also call λ a $\mathbb{Z}_2^n$*-coloring* on P^n here. By $\Lambda(P^n)$ we denote the set of all $\mathbb{Z}_2^n$-colorings on P^n.

Remark 2.1. Since P^n is simple, for each k-face F^k, there are $n-k$ facets $F_{i_1}, ..., F_{i_{n-k}}$ such that $F^k = F_{i_1} \cap \cdots \cap F_{i_{n-k}}$ and $\pi^{-1}(F^k)$ is a transversal intersection of $\pi^{-1}(F_{i_1})$, ..., $\pi^{-1}(F_{i_{n-k}})$. Then the group G_F determined by F^k is actually generated by $\lambda(F_{i_1})$, ..., $\lambda(F_{i_{n-k}})$.

Davis and Januszkiewicz [DJ] gave a reconstruction of M^n by using the $\mathbb{Z}_2^n$-coloring λ and the product bundle $P^n \times \mathbb{Z}_2^n$ over P^n. Geometrically this reconstruction is exactly done by gluing 2^n copies of P^n along their boundaries via λ. Thus this reconstruction can be written as follows:

$$M(\lambda) := P^n \times \mathbb{Z}_2^n/(p,v) \sim (p, v + \lambda(F)) \text{ for } p \in F \in \mathcal{F}(P^n).$$

Then we have

Theorem 2.1 (Davis-Januszkiewicz). *All small covers over P^n are given by $\Gamma(P^n)$= $\{M(\lambda)|\lambda \in \Lambda(P^n)\}$.*

There is a natural action of $\mathrm{GL}(n, \mathbb{Z}_2)$ on $\Lambda(P^n)$ defined by $\lambda \longmapsto \alpha \circ \lambda$, and it is easy to see that such an action is free, and also induces an action of $\mathrm{GL}(n, \mathbb{Z}_2)$ on $\Gamma(P^n)$ by $M(\lambda) \longmapsto M(\alpha \circ \lambda)$. Following [DJ], two small covers $M(\lambda_1)$ and $M(\lambda_2)$ in $\Gamma(P^n)$ are said to be *Davis-Januszkiewicz equivalent* if there is a $\alpha \in \mathrm{GL}(n, \mathbb{Z}_2)$ such that $\lambda_1 = \alpha \circ \lambda_2$. Thus, each Davis-Januszkiewicz equivalence class in $\Gamma(P^n)$ is actually an orbit of the action of $\mathrm{GL}(n, \mathbb{Z}_2)$ on $\Gamma(P^n)$.

2.2. **Betti numbers and h-vector.** The notion of the h-vector plays an essential important role in the theory of polytopes, while the notion of Betti numbers is also so important in the topology of manifolds. Davis-Januszkiewicz theory indicates that the Dehn-Sommerville relations for the h-vectors and the Poincaré duality for the Betti numbers are essentially consistent in the setting of small covers.

Let P^* be the dual of P^n that is a simplicial polytope. Then the boundary ∂P^* denoted by K_P is a finite simplicial complex of dimension $n-1$. For $0 \le i \le n-1$, by f_i one denotes the number of all i-faces in K_P. Then the vector $(f_0, f_1, ..., f_{n-1})$ is called the *f-vector* of P^n, denoted by $\mathbf{f}(P^n)$. Then the *h-vector* denoted by $\mathbf{h}(P^n)$ of P^n is an integer vector $(h_0, h_1, ..., h_n)$ defined from the following equation

$$h_0 t^n + \cdots + h_{n-1} t + h_n = (t-1)^n + f_0 (t-1)^{n-1} + \cdots + f_{n-1}.$$

Theorem 2.2 (Davis-Januszkiewicz). *Let $\pi : M^n \longrightarrow P^n$ be a small cover over a simple convex polytope P^n. Then*

$$\mathbf{h}(P^n) = (h_0, ..., h_n) = (b_0, ..., b_n)$$

where $b_i = \dim H^i(M^n; \mathbb{Z}_2)$.

Remark 2.2. We see from Theorem 2.1 that the Poincaré duality $b_i = b_{n-i}$ agrees with the Dehn-Sommerville relation $h_i = h_{n-i}$.

Example 2.1. For the prism $\mathrm{P}^3(m)$, $\mathbf{h}(\mathrm{P}^3(m)) = (1, m-1, m-1, 1)$, so any small cover over $\mathrm{P}^3(m)$ has the mod 2 Betti numbers $(b_0, b_1, b_2, b_3) = (1, m-1, m-1, 1)$.

2.3. **Stanley-Reisner face ring and equivariant cohomology.** Stanley-Reisner face ring is a basic combinatorial invariant, and equivariant cohomology is an essential invariant in the theory of transformation groups. Davis-Januszkiewicz theory indicates that these two kinds of invariants are also essentially consistent in the setting of small covers.

Let P^n be a simple convex polytope with facet set $\mathcal{F}(P^n) = \{F_1, ..., F_\ell\}$. Following [DJ], the Stanley-Reisner face ring of P^n over $\mathbb{Z}_2$, denoted by $\mathbb{Z}_2(P^n)$, is defined as follows:

$$\mathbb{Z}_2(P^n) = \mathbb{Z}_2[F_1, ..., F_\ell]/I$$

where the F_i's are regarded as indeterminates of degree one, and I is a homogenous ideal generated by all sequence free monomials of the form $F_{i_1}\cdots F_{i_s}$ with $F_{i_1}\cap\cdots\cap F_{i_s}=\emptyset$.

Example 2.2. Let P^n be an n-simplex Δ^n with $n+1$ facets $F_1,...,F_{n+1}$. Then

$$\mathbb{Z}_2(\Delta^n)=\mathbb{Z}_2[F_1,...,F_{n+1}]/(F_1\cdots F_{n+1}).$$

Example 2.3. Let $F_1,...,F_{2n}$ be $2n$ facets of an n-cube I^n with $F_i\cap F_{i+n}=\emptyset, i=1,...,n$. Then

$$\mathbb{Z}_2(I^n)=\mathbb{Z}_2[F_1,...,F_{2n}]/(F_iF_{i+n}|i=1,...,n).$$

Theorem 2.3 (Davis-Januszkiewicz)**.** *Let $\pi: M^n\longrightarrow P^n$ be a small cover over a simple convex polytope P^n. Then its equivariant cohomology*

$$H^*_{\mathbb{Z}_2^n}(M^n;\mathbb{Z}_2)\cong\mathbb{Z}_2(P^n).$$

2.4. **Ordinary cohomology.** Let $\pi: M^n\longrightarrow P^n$ be a small cover over a simple convex polytope P^n with $\mathcal{F}(P^n)=\{F_1,...,F_\ell\}$, and $\lambda:\mathcal{F}(P^n)\longrightarrow\mathbb{Z}_2^n$ its $\mathbb{Z}_2^n$-coloring. Now let us extend $\lambda:\mathcal{F}(P^n)\longrightarrow\mathbb{Z}_2^n$ to a linear map $\widetilde{\lambda}:\mathbb{Z}_2^\ell\longrightarrow\mathbb{Z}_2^n$ by replacing $\{F_1,...,F_\ell\}$ by the basis $\{e_1,...,e_\ell\}$ of $\mathbb{Z}_2^\ell$. Then $\widetilde{\lambda}:\mathbb{Z}_2^\ell\longrightarrow\mathbb{Z}_2^n$ is surjective, and $\widetilde{\lambda}$ can be regarded as an $n\times\ell$-matrix (λ_{ij}), which is written as follows:

$$(\lambda(F_1),...,\lambda(F_\ell)).$$

It is well-known that $H_1(B\mathbb{Z}_2^\ell;\mathbb{Z}_2)=H_1(E\mathbb{Z}_2^n\times_{\mathbb{Z}_2^n}M^n;\mathbb{Z}_2)=\mathbb{Z}_2^\ell$ and $H_1(B\mathbb{Z}_2^n;\mathbb{Z}_2)=\mathbb{Z}_2^n$. So one has that $p_*: H_1(E\mathbb{Z}_2^n\times_{\mathbb{Z}_2^n}M^n;\mathbb{Z}_2)\longrightarrow H_1(B\mathbb{Z}_2^n;\mathbb{Z}_2)$ can be identified with $\widetilde{\lambda}:\mathbb{Z}_2^\ell\longrightarrow\mathbb{Z}_2^n$, where $p: E\mathbb{Z}_2^n\times_{\mathbb{Z}_2^n}M^n\longrightarrow B\mathbb{Z}_2^n$ is the fibration of the Borel construction associating to the universal principal $\mathbb{Z}_2^n$-bundle $E\mathbb{Z}_2^n\longrightarrow B\mathbb{Z}_2^n$. Furthermore, $p^*: H^1(B\mathbb{Z}_2^n;\mathbb{Z}_2)\longrightarrow H^1_{\mathbb{Z}_2^n}(M^n;\mathbb{Z}_2)$ is identified with the dual map $\widetilde{\lambda}^*:\mathbb{Z}_2^{n*}\longrightarrow\mathbb{Z}_2^{\ell*}$, where $\widetilde{\lambda}^*=\widetilde{\lambda}^\top$ as matrices. Therefore, column vectors of $\widetilde{\lambda}^*$ can be understood as linear combinations of $F_1,...,F_\ell$ in the face ring $\mathbb{Z}_2(P^n)=\mathbb{Z}_2[F_1,...,F_\ell]/I$. Write

$$\lambda_i=\lambda_{i1}F_1+\cdots+\lambda_{i\ell}F_\ell.$$

Let J_λ be the homogeneous ideal $(\lambda_1,...,\lambda_n)$ in $\mathbb{Z}_2[F_1,...,F_\ell]$. Davis and Januszkiewicz calculated the ordinary cohomology of M^n, which is stated as follows.

Theorem 2.4 (Davis-Januszkiewicz)**.** *Let $\pi: M^n\longrightarrow P^n$ be a small cover over a simple convex polytope P^n. Then its ordinary cohomology*

$$H^*(M^n;\mathbb{Z}_2)\cong\mathbb{Z}_2[F_1,...,F_\ell]/I+J_\lambda.$$

The following result which will be used later is due to Nakayama and Nishimura [NN].

Proposition 2.5. *Let $\pi: M^n\longrightarrow P^n$ be a small cover over a simple convex polytope P^n, and $\lambda:\mathcal{F}(P^n)=\{F_1,...,F_\ell\}\longrightarrow\mathbb{Z}_2^n$ its $\mathbb{Z}_2^n$-coloring. Then M^n is orientable if and only if there exists an automorphism $\sigma\in\mathrm{GL}(n,\mathbb{Z}_2)$ such that $\lambda'=\sigma\circ\lambda$ satisfies $\sum_{j=1}^n\lambda'_{jl}\equiv 1\mod 2$ for all $1\leq l\leq\ell$, where $\widetilde{\lambda'}=(\lambda'_{ij}):\mathbb{Z}_2^\ell\longrightarrow\mathbb{Z}_2^n$ is the linear extension of λ', as before.*

3. Sector Method

Each point of a simple convex polytope P^n has a neighborhood which is affinely isomorphic to an open subset of $\mathbb{R}^n_{\geq 0}$, so P^n is an n-dimensional nice manifold with corners (see [D]). An automorphism of P^n is a self-homeomorphism of P^n as a manifold with corners, and by $\mathrm{Aut}(P^n)$ we denote the group of automorphisms of P^n. All faces of P^n forms a poset by inclusion. An automorphism of $\mathcal{F}(P^n)$ is a bijection from $\mathcal{F}(P^n)$ to itself which preserves the poset structure of all faces of P^n, and by $\mathrm{Aut}(\mathcal{F}(P^n))$ we denote the group of automorphisms of $\mathcal{F}(P^n)$. Each automorphism of $\mathrm{Aut}(P^n)$ naturally induces an automorphism of $\mathcal{F}(P^n)$. It is well-known (see [BP] or [Z]) that two simple convex polytopes are combinatorially equivalent if and only if they are homeomorphic as manifolds with corners. Thus, the natural homomorphism $\Phi : \mathrm{Aut}(P^n) \longrightarrow \mathrm{Aut}(\mathcal{F}(P^n))$ is surjective.

Definition 3.1. Let P be a simple n-polytope and S be a simple $(n-1)$-polytope. An embedding $i : S \longrightarrow P$ is called a *sector* if the following conditions are satisfied:

(a) $P \backslash i(S)$ have two connected components such that $i(S)$ is the common facet of P_1 and P_2, where P_1, P_2 denote the closures of the two components respectively, called the sub-polytopes of i.
(b) For every face $F \subset S$ of dimension k, $i(F)$ is the subset of a unique $(k+1)$-dimensional face of P^n.

Suppose that $i : S \longrightarrow P$ is a sector with P_1 and P_2 as its two sub-polytopes. Let $i_r : S \to P_r, r = 1, 2$, be the embeddings induced by $p \mapsto i(p)$. Then

$$P = P_1 \coprod P_2 / i_1(s) \sim i_2(s).$$

Furthermore, we can get an induced map $i_* : \mathcal{F}(S) \to \mathcal{F}(P)$, which is poset structure-preserving. Of course, i_* is injective. Now set $\bar{\lambda} := \lambda \circ i_*$, which is called the *derived coloring* of i_* and λ. Obviously, the derived coloring $\bar{\lambda}$ assigns to each facet of S a vector in $\mathbb{Z}_2^n$ and satisfies the independence condition: whenever the intersection of some facets $F_{l_1}, ..., F_{l_r}$ in $\mathcal{F}(S)$ is nonempty, $\bar{\lambda}(F_{l_1}), ..., \bar{\lambda}(F_{l_r})$ are linearly independent. Let (ψ, ρ) be a pair of $\psi \in \mathrm{Aut}(S)$ and $\rho \in \mathrm{GL}(n, \mathbb{Z}_2)$. (ψ, ρ) is called an *auto-equivalence* of S if $\bar{\lambda} \circ \psi = \rho \circ \bar{\lambda}$, where we just abuse φ and the automorphism of $\mathcal{F}(S)$ induced by φ.

Now given a $\mathbb{Z}_2^n$-colored simple n-polytope (P, λ) with a sector $i : S \longrightarrow P$ and its two sub-polytopes P_1, P_2, and fix (ψ, ρ) an auto-equivalence of S. Suppose that P' is another simple n-polytope with $j : S \to P'$ another sector, cutting P' into ${P'}_1$ and ${P'}_2$, such that there are $f_r : P_r \to {P'}_r, r = 1, 2$, which are combinatorially equivalent with $f_1 \circ i_1 = j_1$ and $f_2 \circ i_2 \circ \psi = j_2$. Thus

$$P' = P'_1 \coprod P'_2 / j_1(s) \sim j_2(s) = f_1(P_1) \coprod f_2(P_2) / f_1 \circ i_1(s) \sim f_2 \circ i_2 \circ \psi(s).$$

Remark 3.1. Generally P is not combinatorially equivalent to P' although P_r is combinatorially equivalent to P'_r $(r = 1, 2)$.

Then we can define a $\mathbb{Z}_2^n$-coloring λ' on P' as follows: for each facet $F \in \mathcal{F}(P')$, if $F \cap {P_1}' \neq \emptyset$, then there is a unique facet $F_1 \in \mathcal{F}(P)$ such that $f_1(F_1 \cap P_1) \subset F$. Similarly, if $F \cap {P'}_2 \neq \emptyset$, then there is also a unique facet $F_2 \in \mathcal{F}(P^n)$ such that

$f_2(F_2 \cap P_2) \subset F$. Furthermore, define $\lambda' : \mathcal{F}(P') \to \mathbb{Z}_2^n$ in the following way:

$$\lambda'(F) = \begin{cases} \lambda(F_1) & \text{if } F \cap P_1{}' \neq \emptyset \\ \rho^{-1} \circ \lambda(F_2) & \text{if } F \cap P_2{}' \neq \emptyset. \end{cases}$$

Such λ' is well-defined. In fact, if F has nonempty intersection with both P'_1 and P'_2, then F must lie in the image of j_*, say $F = j_*(f)$ where $f \in \mathcal{F}(S)$. Then from $f_1 \circ i_1 = j_1$ we see that $F_1 = i_*(f)$, and from $f_2 \circ i_2 \circ \psi = j_2$ we see that $F_2 = i_*(\psi(f))$. So $\lambda(F_1) = \bar{\lambda}(f) = \rho^{-1} \circ \bar{\lambda} \circ \psi(f) = \rho^{-1} \circ \lambda \circ i_* \circ \psi(f) = \rho^{-1} \circ \lambda(F_2)$ as desired. In summary, we now have two colored polytopes (P, λ) and (P', λ'). Then using the reconstruction of small covers, we obtain two small covers $M(\lambda) = P \times \mathbb{Z}_2^n/(p, v) \sim (p, v + \lambda(F))$ for $p \in F \in \mathcal{F}(P)$ and $M(\lambda') = P' \times \mathbb{Z}_2^n/(p, v) \sim (p, v + \lambda'(F))$ for $p \in F \in \mathcal{F}(P')$.

Now let us look at two small covers $\pi : M(\lambda) \longrightarrow P$ and $\pi' : M(\lambda') \longrightarrow P'$.

Set $M_r = \pi^{-1}(P_r), r = 1, 2$. Let

$$\mathcal{S} = S \times \mathbb{Z}_2^n/\{(s, v) \sim (s, v + \bar{\lambda}(f)) | s \in f \in \mathcal{F}(S)\}.$$

Then it is easy to see that $\mathcal{S}$ is an $(n-1)$-dimensional closed manifold (but possibly disconnected), called the *sector manifold* here. The sector $i : S \longrightarrow P$ naturally induces an embedding $\iota : \mathcal{S} \hookrightarrow M(\lambda)$ defined by $\{(s, v)\} \longmapsto \{(i(s), v)\}$, and $i_r : S \longrightarrow P_r$ also induces the embedding $\iota_r : \mathcal{S} \hookrightarrow M_r$ $(r = 1, 2)$. Obviously, $\partial M_r = \iota_r(\mathcal{S}) = \pi^{-1}(i(S)) = \iota(\mathcal{S})$. Using this terminology one can write

$$M(\lambda) = M_1 \coprod M_2/\iota_1(x) \sim \iota_2(x) = M_1 \coprod M_2/\{(i_1(s), v)\} \sim \{(i_2(s), v)\},$$

i.e., $M(\lambda)$ is obtained by gluing M_1 and M_2 together along their common boundary $\iota(\mathcal{S})$ via ι_1 and ι_2.

Similarly, set $M'_r = \pi'^{-1}(P'_r), r = 1, 2$. Since P'_r is combinatorially equivalent to P_r for $r = 1, 2$, M'_r is homeomorphic to M_r. More precisely, $f_1 : P_1 \longrightarrow P'_1$ induces an equivariant homeomorphism $\widetilde{f_1} : M_1 \longrightarrow M'_1$ by mapping $x = \{(p, v)\} \longmapsto \{(f_1(p), v)\}$, while $f_2 : P_2 \longrightarrow P'_2$ induces a weakly equivariant homeomorphism $\widetilde{f_2} : M_2 \longrightarrow M'_2$ by mapping $x = \{(p, v)\} \longmapsto \{(f_2(p), \rho^{-1}(v))\}$. Let ι'_r $(r = 1, 2)$ be the embedding $\mathcal{S} \hookrightarrow M'_r$ induced by $j_r : S \longrightarrow P'_r$ and ι' the embedding $\mathcal{S} \hookrightarrow M(\lambda')$ induced by $j : S \longrightarrow P'$. Then $\partial M'_r = \iota'_r(\mathcal{S}) = \pi'^{-1}(i(S)) = \iota'(\mathcal{S})$. Furthermore, one has that

$$M(\lambda') = M'_1 \coprod M'_2/\iota'_1(x) \sim \iota'_2(x) = M'_1 \coprod M'_2/\{(j_1(s), v)\} \sim \{(j_2(s), v)\}.$$

On the other hand, take an arbitrary $v_0 \in \mathbb{Z}_2^n$, using the relation $\bar{\lambda} \circ \psi = \rho \circ \bar{\lambda}$ we see easily that the auto-equivalence (ψ, ρ) of S also induces a weakly equivariant homeomorphism $\Psi : \mathcal{S} \to \mathcal{S}$ defined by $\{(s, v)\} \mapsto \{(\psi(s), \rho(v) + v_0)\}$, so that $\iota_2 \circ \Psi$ gives a new embedding of $\mathcal{S}$ in M_2 and $\widetilde{f_2} \circ \iota_2 \circ \Psi$ also gives a new embedding of $\mathcal{S}$ in M'_2. Let $\widetilde{M}(\lambda) = M_1 \coprod M_2/\iota_1(x) \sim \iota_2(\Psi(x))$. The one has that

Lemma 3.1. *$\widetilde{M}(\lambda)$ is homeomorphic to $M(\lambda')$.*

Proof. Let $z = \{(p, v)\} \in \widetilde{M}(\lambda)$. Define $\Pi : \widetilde{M}(\lambda) \longrightarrow M(\lambda')$ by

$$\Pi(z) = \begin{cases} \widetilde{f_1}(z) & \text{if } z \in M_1 \\ \widetilde{f'}_2(z) & \text{if } z \in M_2 \end{cases}$$

where $\widetilde{f'}_2(z) = \{(f_2(p), \rho^{-1}(v) + \rho^{-1}(v_0))\}$. To show that Π is a homeomorphism, it suffices to prove that for $x \in \mathcal{S}$, if $\iota_1(x) \sim \iota_2(\Psi(x))$ in $\widetilde{M}(\lambda)$, then $\Pi \circ \iota_1(x) \sim \Pi \circ \iota_2(\Psi(x))$ in $M(\lambda')$. Since $\Pi \circ \iota_1 = \widetilde{f}_1 \circ \iota_1 = \iota'_1$, it needs only to check that $\Pi \circ \iota_2 \circ \Psi = \iota'_2$. Let $x = \{(s, v)\}$. Then $\Pi \circ \iota_2 \circ \Psi(x) = \widetilde{f'}_2 \circ \iota_2 \circ \Psi(x) = \widetilde{f'}_2 \circ \iota_2(\{(\psi(s), \rho(v)+v_0)\}) = \widetilde{f'}_2(\{(i_2 \circ \psi(s), \rho(v)+v_0)\} = \{(f_2 \circ i_2 \circ \psi(s), \rho^{-1}(\rho(v)+v_0) + \rho^{-1}(v_0))\} = \{(f_2 \circ i_2 \circ \psi(s), v)\} = \iota'_2(x)$, as desired. □

Remark 3.2. It is easy to see that generally $\widetilde{M}(\lambda)$ is not (weakly) equivariant homeomorphic to $M(\lambda')$ except that ρ is the identity of $\mathbb{Z}_2^n$. Also, clearly the definition of Ψ depends on the auto-equivalence (ψ, ρ) of S and v_0, so we may write Ψ as $\Psi(\psi, \rho, v_0)$ to indicate this dependence.

Now let us consider when $M(\lambda)$ and $M(\lambda')$ are homeomorphic.

Definition 3.2. We say that $\Psi(\psi, \rho, v_0)$ is *extendable* to a self-homeomorphism of M_2 if there is a self-homeomorphism $\widetilde{\Psi} : M_2 \to M_2$ such that $\widetilde{\Psi} \circ \iota_2 = \iota_2 \circ \Psi$.

Theorem 3.2. *If $\Psi(\psi, \rho, v_0)$ is extendable to a self-homeomorphism of M_2, then $M(\lambda)$ is homeomorphic to $M(\lambda')$.*

Proof. Let $\widetilde{\Psi} : M_2 \to M_2$ be a self-homeomorphism with $\widetilde{\Psi} \circ \iota_2 = \iota_2 \circ \Psi$. By Lemma 3.1, one may identify $M(\lambda')$ with $\widetilde{M}(\lambda) = M_1 \coprod M_2/\iota_1(x) \sim \iota_2(\Psi(x))$. Define $H : M(\lambda) \longrightarrow M(\lambda')$ by

$$H(x) = \begin{cases} x & \text{if } x \in M_1 \subset M(\lambda) \\ \widetilde{\Psi}(x) & \text{if } x \in M_2 \subset M(\lambda). \end{cases}$$

An easy observation shows that H is a well-defined homeomorphism. □

Remark 3.3. Take an automorphism $\widetilde{\psi}$ of P_2 such that $\widetilde{\psi}(i_2(S)) = i_2(S)$. Then the restriction $\widetilde{\psi}|_{i_2(S)}$ gives an automorphism of $i_2(S)$. If we can choose (ψ, ρ) with the property that $\psi \circ i_2 = \widetilde{\psi}|_{i_2(S)}$ and $\rho \circ \lambda = \lambda \circ \widetilde{\psi}$, then it is easy to check that $(\widetilde{\psi}, \rho)$ can induce a self-homeomorphism $\widetilde{\Psi}(\widetilde{\psi}, \rho, v_0)$ of M_2, which is defined by $\{(p, v)\} \longmapsto \{(\widetilde{\psi}(p), \rho(v) + v_0)\}$, so that $\widetilde{\Psi}(\widetilde{\psi}, \rho, v_0)$ is an extension of $\Psi(\psi, \rho, v_0)$. In this case, we see that $M(\lambda)$ is homeomorphic to $M(\lambda')$. However, the condition $\rho \circ \lambda = \lambda \circ \widetilde{\psi}$ results in a little bit difficulty for the choice of $(\widetilde{\psi}, \rho)$.

Next let us further analyze $\Psi(\psi, \rho, v_0)$.

Theorem 3.3. *If $\Psi(\psi, \rho, v_0)$ is isotopic to the identity, then $M(\lambda)$ is homeomorphic to $M(\lambda')$.*

Proof. Since the image of the embedding $i : S \to P$ does not contain any vertex of P, we can extend i to an embedding $\widetilde{i} : S \times [-1, 1] \to P$ such that $i = \widetilde{i}(\cdot, 0)$ and each $\widetilde{i}(\cdot, t)$ is a sector. We can further assume that $\widetilde{i}(S \times [-1, 0]) \subset P_1$ and $\widetilde{i}(S \times [0, 1]) \subset P_2$. Now in the world of topology, $\widetilde{i}$ corresponds to an embedding $\widetilde{\iota} : \mathcal{S} \times [-1, 1] \to M(\lambda)$ such that $\iota = \widetilde{\iota}(\cdot, 0)$, and $\widetilde{\iota}(\mathcal{S} \times [0, 1]) \subset M_2$. Now let $\bar{\Psi} : \mathcal{S} \times [0, 1] \to \mathcal{S}$ be an isotopy such that $\bar{\Psi}(\cdot, 0) = \Psi$ and $\bar{\Psi}(\cdot, 1) = \text{id}$. Then define $\widetilde{\Psi} : M_2 \to M_2$ by

$$\widetilde{\Psi}(y) = \begin{cases} \widetilde{\iota}(\bar{\Psi}(x, t), t) & \text{if } y = \widetilde{\iota}(x, t) \in \text{Im}\widetilde{\iota} \\ y & \text{if } y \notin \text{Im}\widetilde{\iota}. \end{cases}$$

One checks easily that $\widetilde{\Psi}$ is a self-homeomorphism of M_2 such that $\widetilde{\Psi}|_{\iota_2(\mathcal{S})} = \iota_2 \circ \Psi$. Moreover, Theorem 3.3 follows by applying Theorem 3.2. □

Definition 3.3. $\Psi(\psi, \rho, v_0)$ is called a *good twist* if $\Psi(\psi, \rho, v_0)$ is isotopic to the identity.

We note that the homeomorphism type of $M(\lambda')$ doesn't depend on the choice of v_0. So to apply Theorem 3.3 we can choose a suitable v_0 such that $\Psi(\psi, \rho, v_0)$ meets the conditions.

Remark 3.4. The sector method above provides a way of how to construct a homeomorphism between two small covers $M(\lambda) \longrightarrow P$ and $M(\lambda') \longrightarrow P'$ regardless of whether P is combinatorially equivalent to P' or not. In addition, the sector method also gives an approach of how to construct a new colored polytope (P', λ') from (P, λ) by the auot-equivalence (ψ, ρ) at the sector $i : S \longrightarrow P$.

4. Application to prisms: Rectangular Sector Method

The objective of this section is to give the application of the sector method to prisms.

Let $\mathrm{P}^3(m)$ denote a 3-dimensional prism that is the product of $[0, 1]$ and an m-gon where $m \geq 3$. When $m \neq 4$, let c, f (the *ceiling* and the *floor*) be the two 2-faces of $\mathrm{P}^3(m)$ that are m-gons. For the 3-cube (i.e., $m = 4$), we specify two opposite 2-faces and distinguish them as ceiling and floor. For convenience, we identify other 2-faces (i.e., side 2-faces) with $s_1, ..., s_m$ in the natural way. Let

$$\Lambda(\mathrm{P}^3(m)) = \{\lambda \big| \lambda \text{ is a } \mathbb{Z}_2^3\text{-coloring on } \mathrm{P}^3(m)\}.$$

4.1. **Rectangular sector.** Generally, any polygon can become a sector in the setting of all 3-polytopes. However, here we shall put attention on rectangular sectors because this will be sufficient enough to the classification of all small covers over prisms.

Throughout the following, choose the rectangle $S = \{(x, y) \in \mathbb{R}^2 \big| |x| \leq 1, |y| \leq 1\}$ in a plane $\mathbb{R}^2$. Clearly S can always be embedded as a sector in any $\mathbb{Z}_2^3$-colored simple 3-polytope (P^3, λ). Fix $\{e_1, e_2, e_3\}$ as a basis of $\mathbb{Z}_2^3$, then it is easy to see that up to Davis-Januszkiewicz equivalence, all possible derived colorings $\bar{\lambda} : \mathcal{F}(S) \longrightarrow \mathbb{Z}_2^3$ and corresponding sector manifolds $\mathcal{S}$ can be stated as follows:

	top edge	left edge	bottom edge	right edge	sector manifold $\mathcal{S}$
$\bar{\lambda}_1$	e_1	e_2	e_1	e_2	union of 2 tori
$\bar{\lambda}_2$	e_1	e_2	$e_1 + e_2$	e_2	union of 2 Klein bottles
$\bar{\lambda}_3$	e_1	e_2	e_3	e_2	torus
$\bar{\lambda}_4$	e_3	e_1	$e_1 + e_3$	e_2	Klein bottle
$\bar{\lambda}_5$	e_3	e_1	$e_1 + e_2 + e_3$	e_2	torus

It is well-known that the symmetric group $\mathrm{Aut}(S)$ of S as a 4-gon is isomorphic to the dihedral group $\mathcal{D}_4$ of order 8, which just contains four reflections. Clearly, each reflection of S may be expressed as a matrix. For example, the reflection along y-axis can be written as $\mathrm{diag}(-1, 1)$, and the reflection along x-axis can be written as $\mathrm{diag}(1, -1)$.

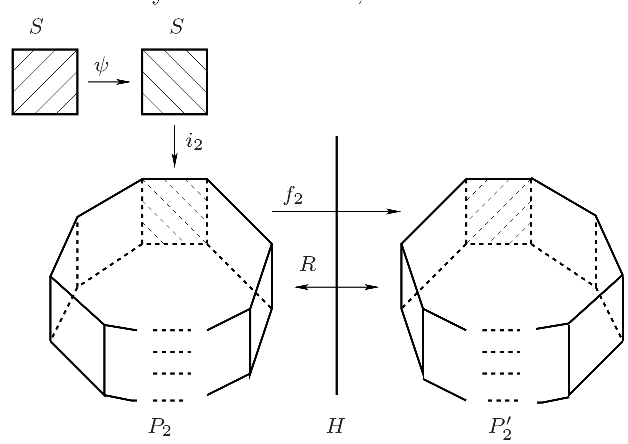
4.2. **Construction of new colored polytopes from** $(\mathrm{P}^3(m), \lambda)$**.** Given a pair $(\mathrm{P}^3(m), \lambda)$ in $\Lambda(\mathrm{P}^3(m))$. We use the convention that all embedded rectangular sectors of $(\mathrm{P}^3(m), \lambda)$ used here are always orthogonal to the ceiling and floor of $\mathrm{P}^3(m)$. Given such a sector $i : S \longrightarrow \mathrm{P}^3(m)$ (note: here we don't need that i must map the top edge and the bottom edge of S into the ceiling and the floor of $\mathrm{P}^3(m)$, respectively), it is easy to see that there are two side faces s_k, s_l ($k \neq l$ and $k < l$) such that $S \longrightarrow \mathrm{P}^3(m)$ is essentially determined by s_k and s_l and it is called the sector *at* s_k, s_l and is denoted by $i(k, l)$, where $S = \{(x, y) \in \mathbb{R}^2 \big| |x| \leq 1, |y| \leq 1\}$. Let P_1 and P_2 be two prisms cut out by $i(k, l) : S \longrightarrow \mathrm{P}^3(m)$ from $\mathrm{P}^3(m)$. Throughout the following, one also uses the convention that P_2 contains side faces $s_k, ..., s_l$ of $\mathrm{P}^3(m)$.

Now, using the sector method we discuss how to construct new colored 3-polytopes (P', λ') from $(\mathrm{P}^3(m), \lambda)$ by auto-equivalences (ψ, ρ) at the sector $i(k, l) : S \longrightarrow \mathrm{P}^3(m)$. For our purpose, we wish that (1) P' is still combinatorially equivalent to $\mathrm{P}^3(m)$, and (2) $\Psi(\psi, \rho, v_0)$ is a good twist, so that each $M(\lambda')$ is homeomorphic to $M(\lambda)$ by Theorem 3.3. This will depend upon the choices of ψ and ρ. Actually, the construction of P' depends upon the choice of ψ, and the definition of λ' depends upon the choice of ρ. In particular, v_0 will provide a convenience for the choices of (ψ, ρ).

Convention: $\psi = \mathrm{diag}(-1, 1)$ or $\mathrm{diag}(1, 1)$ means that $i(k, l)$ maps the top edge and the bottom edge of S into the ceiling and the floor of $\mathrm{P}^3(m)$, respectively, and $\psi = \mathrm{diag}(1, -1)$ means that $i(k, l)$ maps the left edge and the right edge of S into the ceiling and the floor of $\mathrm{P}^3(m)$, respectively.

Lemma 4.1. *If* $\psi = \mathrm{diag}(-1, 1)$ *or* $\mathrm{diag}(1, -1)$ *or* $\mathrm{diag}(1, 1)$*, then we can construct a new polytope* P' *from* $\mathrm{P}^3(m)$ *by* ψ *at the sector* $i(k, l) : S \longrightarrow \mathrm{P}^3(m)$ *such that* P' *is combinatorially equivalent to* $\mathrm{P}^3(m)$*.*

Proof. The case $\psi = \mathrm{diag}(1, 1)$ is trivial. Actually, in this case we can just choose $P'_r = P_r$ and let $f_r : P_r \longrightarrow P'_r$ be the identity, where $r = 1, 2$. If $\psi = \mathrm{diag}(-1, 1)$, to construct P', we first choose $P'_1 = P_1$ and f_1 as the identity from $P_1 \longrightarrow P'_1$, and then choose P'_2 as the image of mirror reflection R of P_2 along a 2-plane H orthogonal to the ceiling and floor of P_2 with $H \cap P_2 = \emptyset$ (i.e., intuitively P'_2 is obtained by reserving the ordering of the side faces $s_k, ..., s_l$ of P_2) and f_2 as the homeomorphism induced by the reflection R, as shown in the following figure.



Now, we clearly see that P' can be defined as $P_1 \coprod P_2'/i_1(s) \sim f_2 \circ i_2 \circ \psi(s)$, which is combinatorially equivalent to $\mathrm{P}^3(m)$. In a similar way, we can prove the case $\psi = \mathrm{diag}(1,-1)$. $\square$

Now suppose that P', which is just constructed from $\mathrm{P}^3(m)$ by a ψ at the sector $i(k,l) : S \longrightarrow \mathrm{P}^3(m)$, is combinatorially equivalent to $\mathrm{P}^3(m)$. Then, as stated in Section 3, we can use ρ to give a coloring λ' on P' as long as ρ satisfies the equation $\rho \circ \bar{\lambda} = \bar{\lambda} \circ \psi$. To guarantee that $M(\lambda')$ is homeomorphic to $M(\lambda)$, we need choose (ψ, ρ) carefully such that $\Psi(\psi, \rho, v_0)$ is a good twist.

Based upon the possible values of $\bar{\lambda}$ (see the table above), we find some good twists and list them as follows:

Sector	(S,λ)	$\Psi(\psi,\rho,v_0)$	ψ	$\rho(e_1)$	$\rho(e_2)$	$\rho(e_3)$	v_0
$S(1)$	$(S,\bar{\lambda}_1)$	$\Psi(\psi_1,\rho_1,v_0^{(1)})$	$\mathrm{diag}(1,-1)$	e_1	e_2	e_3	e_1
$S(2_1)$	$(S,\bar{\lambda}_2)$	$\Psi(\psi_{21},\rho_{21},v_0^{(21)})$	$\mathrm{diag}(1,1)$	e_1	e_2	e_3+e_2	0
$S(2_2)$	$(S,\bar{\lambda}_2)$	$\Psi(\psi_{22},\rho_{22},v_0^{(22)})$	$\mathrm{diag}(1,-1)$	e_1+e_2	e_2	e_3	e_1
$S(3_1)$	$(S,\bar{\lambda}_3)$	$\Psi(\psi_{31},\rho_{31},v_0^{(31)})$	$\mathrm{diag}(-1,1)$	e_1	e_2	e_3	e_2
$S(3_2)$	$(S,\bar{\lambda}_3)$	$\Psi(\psi_{32},\rho_{32},v_0^{(32)})$	$\mathrm{diag}(1,-1)$	e_3	e_2	e_1	e_1
$S(4)$	$(S,\bar{\lambda}_4)$	$\Psi(\psi_4,\rho_4,v_0^{(4)})$	$\mathrm{diag}(1,-1)$	e_1	e_2	e_3+e_1	e_3
$S(5)$	$(S,\bar{\lambda}_5)$	$\Psi(\psi_5,\rho_5,v_0^{(5)})$	$\mathrm{diag}(-1,1)$	e_2	e_1	e_3	e_1

Remark 4.1. It should be pointed out that we have not listed all such good twists. At least we omit compositions of good twists that have already appeared in the above table. However, as we shall see, those good twists listed above are sufficient in the further applications.

Here we only give a detailed argument of $S(1)$ because all other cases can be checked similarly. In $S(1)$, we know that $\mathcal{S} = [-1,1]^2 \times \mathbb{Z}_2^3/\sim$ is the disjoint union of two tori. Then we can write $\mathcal{S} = \{(z_1,z_2,\alpha)\big| z_k \in S^1 \subset \mathbb{C}, k=1,2, \alpha \in \{0,1\}\}$, such that for $\mathbf{x} = \{((x_1,x_2), v = a_1e_2 + a_2e_1 + a_3e_3)\} \in \mathcal{S}$, there is the following one-one correspondence

$$z_k(\mathbf{x}) = \begin{cases} \exp(\mathbf{i}x_k\pi/2) & \text{if } a_k = 0 \\ \exp(\mathbf{i}(\pi - x_k\pi/2)) & \text{if } a_k \neq 0 \end{cases}$$

for $k=1,2$ and $\alpha(\mathbf{x}) = a_3$. Now we consider the map $\Psi(\psi,\rho,v_0)$ where $\psi(x_1,x_2) = (x_1,-x_2)$, $\rho = \mathrm{id}$ and $v_0 = e_1$. An easy computation yields that $\Psi(z_1,z_2,\alpha) = (z_1,-z_2,\alpha)$, which is clearly isotopic to the identity via the homotopy $((z_1, z_2, \alpha),t) \mapsto (z_1, z_2\exp(\mathbf{i}\pi t), \alpha)$ where $t \in [0,1]$. Thus $\Psi(\psi,\rho,v_0)$ is a good twist.

5. Operations on coloring sequences and canonical forms

Now we apply the developed rectangular sector method to study small covers over prisms.

Given a pair $(\mathrm{P}^3(m), \lambda)$ in $\Lambda(\mathrm{P}^3(m))$ and a sector $i(k,l) : S \longrightarrow \mathrm{P}^3(m)$. We have known how to construct a (P', λ') from $(\mathrm{P}^3(m), \lambda)$ by an auto-equivalence (ψ, ρ) at the sector $S \longrightarrow \mathrm{P}^3(m)$. Indeed, by Lemma 4.1, if $\psi = \mathrm{id}$ or $\mathrm{diag}(-1,1)$ or $\mathrm{diag}(1,-1)$, then P' is also a $\mathrm{P}^3(m)$ with the same ceiling and floor coloring, and λ' has the same side coloring sequence as λ on sides faces from s_{l+1} to s_{k-1}. For $k \leq r \leq l$, if $\psi = \mathrm{id}$, then $\lambda'(s_r) = \rho^{-1}\lambda(s_r)$; if $\psi = \mathrm{diag}(-1,1)$ or $\mathrm{diag}(1,-1)$,

$\lambda'(s_r) = \rho^{-1}\lambda(s_{k+l-r})$, that is, we reflect the sequence from s_k to s_l and apply the linear transformation ρ. By Theorem 3.3, when the derived coloring $\bar{\lambda}$ of the sector at s_k, s_l and (ψ, ρ) match a case in the table of last section, we can conclude that $M(\lambda')$ is homeomorphic to $M(\lambda)$. Thus we can reduce $(\mathrm{P}^3(m), \lambda)$ to $(\mathrm{P}^3(m), \lambda')$ without changing the homeomorphism type of the small cover. In this case, both $(\mathrm{P}^3(m), \lambda)$ and $(\mathrm{P}^3(m), \lambda')$ are said to be *sector-equivalent*, denoted by $(\mathrm{P}^3(m), \lambda) \approx (\mathrm{P}^3(m), \lambda')$ or simply $\lambda \approx \lambda'$.

Based upon this rectangular sector method, we shall show that $\Lambda(\mathrm{P}^3(m))$ contains some basic colored pairs, called “canonical forms”, such that any pair in $\Lambda(\mathrm{P}^3(m))$ is sector-equivalent to one of canonical forms. This means that up to homeomorphism, those canonical forms determine all small covers over $\mathrm{P}^3(m)$.

For a convenience, after fixing the colorings of ceiling and floor, we use the *convention* that a coloring on $\mathrm{P}^3(m)$ will simply be described as a sequence by writing its side face colorings in order, keeping in mind that the first one is next to the last.

Definition 5.1. A coloring $\lambda \in \lambda(\mathrm{P}^3(m))$ is said to be *2-independent* if all $\lambda(s_i), i = 1, ..., m$, span a 2-dimensional subspace of $\mathbb{Z}_2^3$; otherwise it's said to be *3-independent*. If $\lambda(c) = \lambda(f)$, then λ is said to be *trivial*; otherwise *nontrivial*.

The argument is divided into two cases: (i) λ is trivial; (ii) λ is nontrivial.

5.1. **Trivial colorings.** Given a pair $(\mathrm{P}^3(m), \lambda)$ in $\Lambda(\mathrm{P}^3(m))$, throughout the following suppose that λ is trivial with $\lambda(c) = \lambda(f) = e_1$. Let $\{\alpha, \beta, e_1\}$ be a basis of $\mathbb{Z}_2^3$, and let $\gamma = \alpha + \beta$. Write $\bar{\alpha} = \alpha + e_1$, $\bar{\beta} = \beta + e_1$ and $\bar{\gamma} = \gamma + e_1$. We say that λ satisfies the property $(\star)$ if all three letters α, β, γ (with or without bar we don't care) appear in its coloring sequence.

Applying sectors $S(1), S(2_1), S(2_2)$ and $S(3_2)$ to the trivial coloring λ gives the following four fundamental operations on its coloring sequence $(\lambda(s_1), ..., \lambda(s_m))$:

- O_1 Take two side faces s_k, s_l $(k < l)$ with the same coloring and then use $S(1)$ to reflect the coloring sequence of $s_k, s_{k+1}, ..., s_l$.
- O_{21} Take two faces s_k, s_l $(k < l)$ with $\lambda(s_k) = \lambda(s_l) + \lambda(c)$ (without loss of generality, assume that $\{\lambda(s_k), \lambda(s_l)\} = \{\alpha, \bar{\alpha}\}$), and then by using $S(2_1)$, we can do a linear transform $(e_1, \alpha, \beta) \mapsto (e_1, \alpha, \bar{\beta})$ to change the coloring sequence of $s_k, s_{k+1}, ..., s_l$.
- O_{22} Take two faces s_k, s_l $(k < l)$ with $\lambda(s_k) = \lambda(s_l) + \lambda(c)$ and $\{\lambda(s_k), \lambda(s_l)\} = \{\alpha, \bar{\alpha}\}$ as above, then by using $S(2_2)$ we can reflect the coloring sequence of $s_k, s_{k+1}, ..., s_l$ and do a linear transform $(e_1, \alpha, \beta) \mapsto (e_1, \bar{\alpha}, \beta)$ to change the reflected coloring sequence.
- O_{32} Take s_k, s_l $(k < l)$ with $\lambda(s_k), \lambda(s_l), e_1$ independent, and then by using $S(3_2)$ we can reflect the coloring sequence of $s_k, s_{k+1}, .., s_l$ and do a linear transform $(\lambda(s_k), \lambda(s_l), e_1) \mapsto (\lambda(s_l), \lambda(s_k), e_1)$ to change the reflected coloring sequence.

Lemma 5.1. *The trivial coloring λ with the property $(\star)$ is always sector-equivalent to a coloring whose coloring sequence contains only one of both γ and $\bar{\gamma}$.*

Proof. Let $\widetilde{\gamma} = \gamma$ or $\bar{\gamma}$. With no loss, assume that the time number ℓ of $\widetilde{\gamma}$ appearing in the coloring sequence of λ is greater than one. Up to Davis-Januszkiewicz equivalence, one also may assume that $\ell < m/2$. By the definition of λ, it is easy

to see that any two $\widetilde{\gamma}$'s in the coloring sequence cannot become neighbors. Let $\widetilde{\gamma}, x_1, ..., x_r, \widetilde{\gamma}, y$ with $x_i, y \neq \widetilde{\gamma}$ be a subsequence of the coloring sequence. If $r > 1$, we proceed as follows:

(1) When $x_1 = y$, by doing the operation O_1 on $x_1, ..., x_r, \widetilde{\gamma}, y$, we may only change the subsequence $\widetilde{\gamma}, x_1, ..., x_r, \widetilde{\gamma}, y$ into $\widetilde{\gamma}, y, \widetilde{\gamma}, x_r, ..., x_1$ in the coloring sequence, and the value of ℓ is unchanged.
(2) When $x_1 - y = e_1$, with no loss one may assume that $x_1 = \alpha, y = \bar{\alpha}$. Then by doing the operation O_{22} on $x_1 = \alpha, x_2, ..., x_r, \widetilde{\gamma}, y = \bar{\alpha}$, we may only change the subsequence $\widetilde{\gamma}, \alpha, x_2, ..., x_r, \widetilde{\gamma}, \bar{\alpha}$ into $\widetilde{\gamma}, \alpha, \widetilde{\gamma}, x'_r, ..., x'_2, \bar{\alpha}$ with $x'_i \neq \widetilde{\gamma}$, and the value of ℓ is unchanged.
(3) When x_1, y, e_1 are linearly independent, with no loss one may assume that $x_1 = \alpha, y = \beta$. Then by doing the operation O_{32} on $x_1 = \alpha, x_2, ..., x_r, \widetilde{\gamma}, y = \beta$, we may only change the subsequence $\widetilde{\gamma}, \alpha, x_2, ..., x_r, \widetilde{\gamma}, \beta$ into $\widetilde{\gamma}$, α, $\widetilde{\gamma}, x'_r, ..., x'_2, \beta$ with $x'_i \neq \widetilde{\gamma}$, and the value of ℓ is unchanged.

Thus, we may reduce the coloring λ to another coloring with the following coloring sequence

$$(\widetilde{\gamma}, y_1, \widetilde{\gamma}, y_2, ..., \widetilde{\gamma}, y_{\ell-1}, \widetilde{\gamma}, y_\ell, z_1, ..., z_{m-2\ell}) \text{ with } m - 2\ell > 0. \tag{5.1}$$

Without loss of generality, one may assume that $y_{\ell-1} = \alpha$. If $y_\ell = \beta$ or $\bar{\beta}$, by doing the operation O_{32} on $\widetilde{\gamma}, y_{\ell-1}, \widetilde{\gamma}, y_\ell$, one may change $\widetilde{\gamma}, y_{\ell-1}, \widetilde{\gamma}, y_\ell$ into $\widetilde{\gamma}, y_\ell, y_{\ell-1}, y_\ell$, so that the coloring sequence (5.1) is reduced to ($\widetilde{\gamma}$, y_1, $\widetilde{\gamma}$, $y_2, ...,$ $\widetilde{\gamma}, y_{\ell-2}$, $\widetilde{\gamma}$, $y_\ell, y_{\ell-1}, y_\ell$, $z_1, ..., z_{m-2\ell}$). If $y_\ell = \alpha$ or $\bar{\alpha}$, then $z_1 = \beta$ or $\bar{\beta}$. By doing the operation O_{32} on $\widetilde{\gamma}, y_{\ell-1}, \widetilde{\gamma}, y_\ell, z_1$, one may change $\widetilde{\gamma}, y_{\ell-1}, \widetilde{\gamma}, y_\ell, z_1$ into $\widetilde{\gamma}, y_\ell, z_1, y_{\ell-1}, z_1$, so that the coloring sequence (5.1) is reduced to ($\widetilde{\gamma}$, y_1, $\widetilde{\gamma}$, $y_2, ...,$ $\widetilde{\gamma}, y_{\ell-2}$, $\widetilde{\gamma}$, $y_\ell, z_1, y_{\ell-1}$, $z_1, ..., z_{m-2\ell}$). So we have managed to reduce the number ℓ of $\widetilde{\gamma}$'s by 1. We can continue this process until we reach $\ell = 1$, as desired. □

Now let us determine the "canonical form" of the trivial coloring λ on $\mathrm{P}^3(m)$.

First let us consider the case in which λ is 2-independent

Proposition 5.2. *Suppose that λ is 2-independent. Then*

(1) *If λ doesn't possess the property ($\star$), then m is even and λ is sector-equivalent to the canonical form λ_{C_1} with the coloring sequence $\mathcal{C}_1 = (\alpha,$ $\beta, ..., \alpha, \beta)$.*
(2) *If λ possesses the property ($\star$), then λ is sector-equivalent to one of the following two canonical forms:* (a) *λ_{C_2} with m even and with the coloring sequence $\mathcal{C}_2 = (\alpha, \gamma, \alpha, \beta, ..., \alpha, \beta)$;* (b) *$\lambda_{C_3}$ with m odd and with the coloring sequence $\mathcal{C}_3 = (\alpha, \gamma, \beta, \alpha, \beta, ..., \alpha, \beta)$.*

Proof. If λ doesn't possess the property ($\star$), then it is easy to see that λ is unique up to Davis-Januszkiewicz equivalence and m is even. So Proposition 5.2(1) follows from this. By Lemma 5.1, an easy observation shows that Proposition 5.2(2) holds. □

Next let us consider the case in which λ is 3-independent.

Proposition 5.3. *If λ is 3-independent without the property ($\star$), then m is even and λ is sector-equivalent to the canonical form λ_{C_4} with the coloring sequence $\mathcal{C}_4 = (\bar{\alpha}, \beta, \alpha, \beta, ..., \alpha, \beta)$.*

Proof. Without loss of generality assume that each element in the coloring sequence $\mathcal{C}$ of λ is in the set $\{\alpha, \beta, \bar{\alpha}, \bar{\beta}\}$ and that both α and $\bar{\alpha}$ must appear in $\mathcal{C}$. Since α and $\bar{\alpha}$ (or β and $\bar{\beta}$) can become neighbors, one has that m must be even.

Similarly to the argument of Lemma 5.1, by using the operations O_1 and O_{22}, we may reduce λ to a coloring with the coloring sequence

$$(\bar{\alpha}, x_1, ..., \bar{\alpha}, x_r, \alpha, x_{r+1}, ..., \alpha, x_{\frac{m}{2}}) \tag{5.2}$$

where $r \geq 1$ and $x_i = \beta$ or $\bar{\beta}$, and this reduction doesn't change the number of bars on α's. If $r > 1$, by doing the operation O_{22} on $\bar{\alpha}, x_{r-1}, \bar{\alpha}, x_r, \alpha$, we may reduce the sequence (5.2) to

$$(\bar{\alpha}, x_1, ..., \bar{\alpha}, x_{r-2}, \bar{\alpha}, x_r, \alpha, x_{r-1}, \alpha, x_{r+1}, ..., \alpha, x_{\frac{m}{2}}),$$

reducing the number of bars on α's by one. This process can be carried out until the sequence (5.2) is reduced to

$$(\bar{\alpha}, x_r, \alpha, x_1, ..., \alpha, x_{r-2}, \alpha, x_{r-1}, \alpha, x_{r+1}, ..., \alpha, x_{\frac{m}{2}}). \tag{5.3}$$

Next, we claim that by using the operation O_{21}, we may remove all possible bars on β's in the sequence (5.3). In fact, if $x_r = \bar{\beta}$, then applying the operation O_{21} on $\bar{\alpha}, x_r, \alpha$, we may remove the bar on x_r. Generally, with no loss, one may assume that $x_j = \bar{\beta}$ and $x_r = x_l = \beta$ where $l \in \{1, ..., j-1\}$ if $j \leq r+1$ and $j \neq r$, and $l \in \{1, ..., r-1, r+1, ..., j-1\}$ if $j > r+1$. Applying the operation O_{21} on $\bar{\alpha}, x_r, ..., \alpha, x_j = \bar{\beta}, \alpha$, one may remove the bar on $x_j = \bar{\beta}$, but add the bar on x_r and x_l's. Again applying the operation O_{21} on $\bar{\alpha}, \bar{x}_r, ..., \alpha, \bar{x}_l, ..., \alpha, \bar{x}_{j-1}, \alpha$, one may remove all bars on $\bar{x}_r$ and $\bar{x}_l$'s. Thus, by carrying on this procedure, the above claim holds. This completes the proof. □

Proposition 5.4. *If λ is 3-independent with the property $(\star)$ and $m > 4$, then λ is sector-equivalent to one of the following six canonical forms:*

(1) λ_{C_5} *with m odd and with the coloring sequence $\mathcal{C}_5 = (\bar{\gamma}, \alpha, \beta, ..., \alpha, \beta)$.*
(2) λ_{C_6} *with m odd and with the coloring sequence $\mathcal{C}_6 = (\bar{\gamma}, \bar{\alpha}, \beta, \alpha, \beta, ..., \alpha, \beta)$.*
(3) λ_{C_7} *with m odd and with the coloring sequence $\mathcal{C}_7 = (\gamma, \bar{\alpha}, \beta, \alpha, \beta, ..., \alpha, \beta)$.*
(4) λ_{C_8} *with m even and with the coloring sequence $\mathcal{C}_8 = (\bar{\alpha}, \gamma, \bar{\alpha}, \beta, \alpha, \beta, ..., \alpha, \beta)$.*
(5) λ_{C_9} *with m even and with the coloring sequence $\mathcal{C}_9 = (\alpha, \gamma, \bar{\alpha}, \beta, \alpha, \beta, ..., \alpha, \beta)$.*
(6) $\lambda_{C_{10}}$ *with m even and with the coloring sequence $\mathcal{C}_{10} = (\alpha, \bar{\gamma}, \alpha, \beta, ..., \alpha, \beta)$.*

Proof. By Lemma 5.1, one may assume that $\widetilde{\gamma}(= \gamma$ or $\bar{\gamma})$ appears only one time in the coloring sequence $\mathcal{C}$ of λ.

Case (I): *m is odd.* If only one of both α and $\bar{\alpha}$ appears in $\mathcal{C}$ and the same thing also happens for both β and $\bar{\beta}$, then $\widetilde{\gamma}$ must be $\bar{\gamma}$ so $\mathcal{C}$ is just $\mathcal{C}_5$. Otherwise we can carry out our argument as in Proposition 5.3 on the subsequence in $\mathcal{C}$ of containing no $\widetilde{\gamma}$, so that $\mathcal{C}$ is sector-equivalent to $\mathcal{C}_6$ or $\mathcal{C}_7$.

Case (II): *m is even.* Consider two neighbors of $\widetilde{\gamma}$, since m is even, such two neighbors must be the same letter. Up to Davis-Januszkiewicz equivalence, one may assume that they are $\{\alpha, \alpha\}$ or $\{\alpha, \bar{\alpha}\}$.

If two neighbors of $\widetilde{\gamma}$ are $\{\alpha, \bar{\alpha}\}$, with no loss assume that $\mathcal{C} = (\alpha, \widetilde{\gamma}, \bar{\alpha}, x_4, ..., x_m)$. Then we can carry out our argument as in Proposition 5.3 on $\bar{\alpha}, x_4, ..., x_m$, so that $\mathcal{C}$ may be reduced to $\mathcal{C}' = (\alpha, \widetilde{\gamma}, \bar{\alpha}, \beta, \alpha, \beta, ..., \alpha, \beta)$. If $\widetilde{\gamma} = \gamma$, then $\mathcal{C}'$ is just $\mathcal{C}_8$. If $\widetilde{\gamma} = \bar{\gamma}$, applying the operation O_{22} on $\alpha, \bar{\gamma}, \bar{\alpha}$, one may further reduce $\mathcal{C}'$ to $\mathcal{C}_9$.

Now suppose that two neighbors of $\widetilde{\gamma}$ are $\{\alpha, \alpha\}$. If $\mathcal{C}$ only contains α and β except for $\widetilde{\gamma}$, then $\mathcal{C}$ is just $\mathcal{C}_{10}$. Otherwise, with no loss assume that $\mathcal{C}$ also contains $\bar{\alpha}$. Up to Davis-Januszkiewicz equivalence, by using the linear transformation $(e_1, \alpha, \beta) \longmapsto (e_1, \bar{\alpha}, \beta)$ one may write $\mathcal{C} = (\bar{\alpha}, \widetilde{\gamma}, \bar{\alpha}, x_4, ..., x_m)$. Furthermore, Then we can carry out our argument as in Proposition 5.3 to reduce $\mathcal{C}$ to $(\bar{\alpha}, \widetilde{\gamma}, \bar{\alpha}, \beta, \alpha, \beta, ..., \alpha, \beta)$. If $\widetilde{\gamma}$ is not γ, applying the operation O_{22} on $\alpha, \bar{\gamma}, \bar{\alpha}$, one may further reduce $\mathcal{C}'$ to $\mathcal{C}_8$. □

Remark 5.1. An easy observation shows that for a 3-independent trivial coloring λ with the property $(\star)$, if $m = 3$, then λ is just sector-equivalent to the following canonical form

$$\lambda_{C^3} \text{ with the coloring sequence } \mathcal{C}^3 = (\bar{\gamma}, \alpha, \beta)$$

and if $m = 4$, then λ is just sector-equivalent to one of the following two canonical forms

(1) $\lambda_{C_1^4}$ with the coloring sequence $\mathcal{C}_1^4 = (\alpha, \gamma, \bar{\alpha}, \beta)$.
(2) $\lambda_{C_2^4}$ with the coloring sequence $\mathcal{C}_2^4 = (\alpha, \bar{\gamma}, \alpha, \beta)$.

Combining Propositions 5.2-5.4 and Remark 5.1 gives the following

Corollary 5.5. *The number of homeomorphism classes of small covers over* $\mathrm{P}^3(m)$ *with trivial colorings is at most*

$$N_t(m) = \begin{cases} 2 & \text{if } m = 3 \\ 4 & \text{if } m > 3 \text{ is odd} \\ 6 & \text{if } m \text{ is even} \end{cases}$$

By Proposition 2.5, a direct observation shows that

Corollary 5.6. $M(\lambda_{C_i}), i = 1, 5, 10$, *are orientable, and* $M(\lambda_{C_i}), i = 2, 3, 4, 6, 7, 8, 9$, *are non-orientable.*

5.2. **Nontrivial Prism Small Covers.** Given a pair $(\mathrm{P}^3(m), \lambda)$ in $\Lambda(\mathrm{P}^3(m))$, throughout suppose that λ is nontrivial, i.e., $\lambda(c) \neq \lambda(f)$.

Definition 5.2. Let $M_\lambda = \{s_i | \mathrm{Span}\{\lambda(s_{i-1}), \lambda(s_i), \lambda(s_{i+1})\} = \mathbb{Z}_2^3\}$ and let $m_\lambda := |M_\lambda|$ denote the number of side faces in M_λ. Set $\lambda_0 := \lambda(c) - \lambda(f)$. Let $N_\lambda = \{s_i | \lambda(s_i) = \lambda_0\}$, and let $n_\lambda := |N_\lambda|$ denote the number of side faces in N_λ.

Lemma 5.7. *Let λ be a nontrivial coloring on* $\mathrm{P}^3(m)$ *with $m > 3$. Then*

(1) $m_\lambda \leq n_\lambda \leq m/2$. *In particular, if m is odd, then $n_\lambda > 0$.*
(2) m_λ *is even.*

Proof. First, $n_\lambda \leq m/2$ is obvious since any two faces in N_λ are not adjacent. To show that $m_\lambda \leq n_\lambda$, take one $s_i \in N_3$. Then $\lambda(s_{i-1}), \lambda(s_i), \lambda(s_{i+1})$ are linearly independent. Furthermore, the linear independence of $\{\lambda(s_{i-1}), \lambda(s_i), \lambda(c)\}$ and $\{\lambda(s_i), \lambda(s_{i+1}), \lambda(c)\}$ implies that $\lambda(c)$ must be either $\lambda(s_{i-1}) + \lambda(s_{i+1})$ or $\lambda(s_{i-1}) + \lambda(s_i) + \lambda(s_{i+1})$. This is also true for $\lambda(f)$. Now $\lambda(c) \neq \lambda(f)$ makes sure that $\lambda(c) - \lambda(f) = \lambda(s_i)$, so that $s_i \in N_\lambda$. Thus, $M_\lambda \subseteq N_\lambda$, i.e., $m_\lambda \leq n_\lambda$. Moreover, if $n_\lambda = 0$ then $m_\lambda = 0$, so that the coloring sequence of λ is 2-independent. However, $n_\lambda = 0$ means that one has only two choices of colors. This forces m to be even.

With no loss, assume that $m_\lambda > 0$ (since 0 is even). For each i, let V_i denote the subspace spanned by $\lambda(s_i)$ and $\lambda(s_{i+1})$. Obviously, if $s_i \in M_\lambda$ then $V_i \neq V_{i-1}$, and

if $s_i \notin M_\lambda$ then $V_i = V_{i-1}$. Thus, $s_i \in M_\lambda$ if and only if $V_i \neq V_{i-1}$. Next we claim that for any i, $\lambda_0 \in V_i$. In fact, if $s_i \in M_\lambda$, since $M_\lambda \subseteq N_\lambda$, we have that $s_i \in N_\lambda$ so $\lambda_0 = \lambda(s_i) \in V_i$. If $s_i \notin M_\lambda$ then $V_i = V_{i-1}$. Since $V_i = V_{i-1}$ contains no $\lambda(c)$ and $\lambda(f)$, λ_0 must be in $V_i = V_{i-1}$. This proves the claim. Furthermore, for any i, V_i must be either $\mathrm{Span}\{\lambda_0, \alpha\}$ or $\mathrm{Span}\{\lambda_0, \lambda(c)+\alpha\}$, where α is a nonzero element such that α, $\lambda(c)$ and $\lambda(f)$ are linearly independent. Now we clearly see that there is a switch of choosing either $\mathrm{Span}\{\lambda_0, \alpha\}$ or $\mathrm{Span}\{\lambda_0, \lambda(c)+\alpha\}$ exactly when we pass $s_i \in M_\lambda$. But the total number of switches must be even. So m_λ is even. □

Remark 5.2. An easy observation shows that if $m = 3$, then there is a possibility that $m_\lambda = 3$ but still $n_\lambda = 1 < 3/2$. This is exactly an exception only for m_λ in the case $m = 3$.

Throughout the following, assume that $m > 3$.

Applying sectors $S(2_1), S(3_1), S(4)$ and $S(5)$ to the nontrivial coloring λ gives the following four fundamental operations on its coloring sequence:

- $\bar{\mathrm{O}}_{21}$ Take $s_k, s_l \in N_\lambda$ with $k < l$, by using $S(2_1)$, we may do a linear transformation $(\lambda(c), \lambda_0, \lambda(s_{k+1}) \mapsto (\lambda(c), \lambda_0, \lambda(s_{k+1}) + \lambda_0)$ to change the coloring sequence of $s_k, s_{k+1}, ..., s_l$.
- O_{31} Take s_k, s_l $(k < l)$ with $\lambda(s_k) = \lambda(s_l) \neq \lambda_0$, and use $S(3_1)$ to reflect the coloring sequence of $s_k, s_{k+1}, ..., s_l$.
- O_4 Take s_k, s_l $(k < l)$ with $\lambda(s_k) - \lambda(s_l) = \lambda(c)$ or $\lambda(f)$, we use $S(4)$ to reflect the coloring sequence of $s_k, s_{k+1}, ..., s_l$ and then to do a linear transformation $(\lambda_0, \lambda(s_k), \lambda(s_l)) \mapsto (\lambda_0, \lambda(s_l), \lambda(s_k))$ to change the reflected coloring sequence.
- O_5 Take s_k, s_l $(k < l)$ with $\lambda(s_k) - \lambda(s_l) = \lambda_0$, we use $S(4)$ to reflect the coloring sequence of $s_k, s_{k+1}, ..., s_l$ and then to do a linear transformation $(\lambda(c), \lambda(s_k),\ \lambda(s_l)) \mapsto (\lambda(c), \lambda(s_l), \lambda(s_k))$ to change the reflected coloring sequence.

It is easy to check the following

Lemma 5.8. *The operations* $\bar{\mathrm{O}}_{21}$, O_{31}, O_4 *and* O_5 *above will not change* m_λ, n_λ *of the nontrivial coloring* λ.

Without loss of generality, throughout the following one assumes that $\lambda(c) = e_1, \lambda(f) = e_1 + e_2$, so that $\lambda_0 = e_2$, where $\{e_1, e_2, e_3\}$ is a basis of $\mathbb{Z}_2^3$.

Proposition 5.9. *For* $m > 3$, *each nontrivial coloring* λ *is sector-equivalent to the following canonical form* λ_{C_*} *with the coloring sequence*

$$\mathcal{C}_* = (e_2, x_1, e_2, ..., e_2, x_{m_\lambda}, e_2, y_1, ..., e_2, y_{n_\lambda - m_\lambda}, z_1, ..., z_{m-2n_\lambda}) \tag{5.4}$$

where $x_i = \begin{cases} e_1 + e_3 & \text{if } i \text{ is odd} \\ e_3 & \text{if } i \text{ is even} \end{cases}$ *and for all* $1 \leq i \leq n_\lambda - m_\lambda$, $y_i = e_3$ *and* $z_i = \begin{cases} e_2 + e_3 & \text{if } i \text{ is odd} \\ e_3 & \text{if } i \text{ is even.} \end{cases}$

Proof. If $n_\lambda \leq 1$ then clearly the coloring λ can be reduced to a coloring with the coloring sequence

$$\begin{cases} (e_2, e_3, e_2+e_3, e_3, ..., e_2+e_3, e_3) & \text{if } n_\lambda = 1 \text{ and } m \text{ is even} \\ (e_2, e_3, e_2+e_3, e_3, ..., e_2+e_3, e_3, e_2+e_3) & \text{if } n_\lambda = 1 \text{ and } m \text{ is odd} \\ (e_2+e_3, e_3, ..., e_2+e_3, e_3). & \text{if } n_\lambda = 0 \end{cases}$$

If $n_\lambda \geq 2$, we may choose two s_k and s_l in N_λ with $k < l$. Consider the coloring sub-sequence

$$(\lambda(s_k) =)e_2, r_1, ..., r_{l-2}, e_2(= \lambda(s_l)), r_{l-1} \tag{5.5}$$

of $s_k, ..., s_l, s_{l+1}$, it is easy to see that $r_1 - r_{l-1} \in \text{Span}\{e_1, e_2\}$. Then when when $r_1 - r_{l-1} = 0$ (resp. e_1 or $e_1 + e_2$, e_2), we may do the operation O_{31} (resp. O_4 or O_5) on $r_1, ..., r_{l-2}, e_2, r_{l-1}$ from s_{k+1} to s_{l+1}, and change (5.5) into $e_2, r_{l-1}, e_2, r'_{l-2}, ..., r'_2, r_1$. With this understood, assume that $N_\lambda = \{s_1, s_3, ..., s_{2n_\lambda-1}\}$, so we may write the coloring sequence of λ as follows:

$$\mathcal{C} = (e_2, \alpha_1, ..., e_2, \alpha_{n_\lambda}, \beta_1, ..., \beta_{m-2n_\lambda})$$

with $\alpha_{n_\lambda}, \beta_i \in \{e_2 + e_3, e_3\}$. By doing the operation $\bar{\mathrm{O}}_{21}$ on $\alpha_{n_\lambda}, \beta_1, ..., \beta_{m-2n_\lambda}$, we may reduce $\mathcal{C}$ to $\mathcal{C}' = (e_2, \alpha_1, ..., \alpha_{n_\lambda-1}, e_2, e_3, z_1, ..., z_{m-2n_\lambda})$ such that z_i is $e_2 + e_3$ if i is odd, and e_3 if i is even. Then we may further use the operation O_4 to reduce $\mathcal{C}'$ to $\mathcal{C}''$ with $M_\lambda = \{s_1, ..., s_{2m_\lambda-1}\}$ and without changing the part $\mathcal{C}' - \{e_2, \alpha_1, ..., \alpha_{n_\lambda-1}, e_2\}$. Finally, by using the operation $\bar{\mathrm{O}}_{21}$, we may reduce $\mathcal{C}''$ to $\mathcal{C}_*$ as desired. $\square$

Together with Theorem 3.3, Lemmas 5.7-5.8 and Proposition 5.9, it easily follows that

Corollary 5.10. *Let λ_1, λ_2 be two nontrivial colorings on $\mathrm{P}^3(m)$ with $m > 3$. If $(m_{\lambda_1}, n_{\lambda_1}) = (m_{\lambda_2}, n_{\lambda_2})$, then $M(\lambda_1)$ and $M(\lambda_2)$ are homeomorphic.*

Corollary 5.11. *For $m > 3$, let (k, l) be a pair such that (1) $l \leq k \leq m/2$ and if $2 \nmid m$ then $k > 0$; and (2) l is even. Then there is a nontrivial coloring λ on $\mathrm{P}^3(m)$ with $(n_\lambda, m_\lambda) = (k, l)$.*

As a consequence of Proposition 2.5 and Proposition 5.9, one also has

Corollary 5.12. *Let λ be a nontrivial coloring on $\mathrm{P}^3(m)$ with $m > 3$. Then $M(\lambda)$ is orientable if $n_\lambda = 0$, and non-orientable if $n_\lambda > 0$.*

6. Mod 2 cohomology rings and two invariants

Given a pair $(\mathrm{P}^3(m), \lambda)$ in $\Lambda(\mathrm{P}^3(m))$, one knows that the mod 2 cohomology ring of $M(\lambda)$ is

$$H^*(M(\lambda); \mathbb{Z}_2) = \mathbb{Z}_2[c, f, s_1, ..., s_m]/I + J_\lambda$$

where I is the ideal generated by cf and $s_i s_j$ with $s_i \cap s_j = \emptyset$, and J_λ is the ideal generated by three linear relations (determined by the $3 \times (m + 2)$ matrix $(\lambda(c), \lambda(f), \lambda(s_1), ..., \lambda(s_m))$).

6.1. **Two invariants $\Delta(\lambda)$ and $\mathcal{B}(\lambda)$.** Now let us introduce two invariants in $H^*(M(\lambda);\mathbb{Z}_2)$. Set

$$\mathcal{H}^1_\lambda = \{x \in H^1(M(\lambda);\mathbb{Z}_2) \big| x^2 = 0\}$$
$$\mathcal{H}^2_\lambda = \{x^2 \big| x \in H^1(M(\lambda);\mathbb{Z}_2)\}$$

and

$$\mathcal{K}_\lambda = \mathrm{Span}\{xy \big| x \in H^1(M(\lambda);\mathbb{Z}_2), y \in \mathcal{H}^1_\lambda\}.$$

Clearly, they are all vector spaces over $\mathbb{Z}_2$, and

$$\dim \mathcal{H}^2_\lambda = \dim H^1(M(\lambda);\mathbb{Z}_2) - \dim \mathcal{H}^1_\lambda = m - 1 - \dim \mathcal{H}^1_\lambda.$$

Note that $\dim H^1(M(\lambda);\mathbb{Z}_2) = m-1$ by Example 2.1.

Obviously, $\dim \mathcal{H}^1_\lambda$ is an invariant of the cohomology ring $H^*(M(\lambda);\mathbb{Z}_2)$, denoted by $\Delta(\lambda)$.

Define a bilinear map

$$\omega : H^1(M(\lambda);\mathbb{Z}_2) \times \mathcal{H}^1_\lambda \longrightarrow \mathcal{K}_\lambda/(\mathcal{K}_\lambda \cap \mathcal{H}^2_\lambda)$$

by $(x, y) \longmapsto [xy]$, which is surjective. Let $\mathrm{Hom}(\mathcal{K}_\lambda/(\mathcal{K}_\lambda \cap \mathcal{H}^2_\lambda), \mathbb{Z}_2)$ be the dual space of $\mathcal{K}_\lambda/(\mathcal{K}_\lambda \cap \mathcal{H}^2_\lambda)$. Take a $\theta \in \mathrm{Hom}(\mathcal{K}_\lambda/(\mathcal{K}_\lambda \cap \mathcal{H}^2_\lambda), \mathbb{Z}_2)$, one can obtain a bilinear map

$$\theta \circ \omega : H^1(M(\lambda);\mathbb{Z}_2) \times \mathcal{H}^1_\lambda \longrightarrow \mathbb{Z}_2,$$

which corresponds an $(m-1)\times\Delta(\lambda)$-matrix. Let $b_r(\lambda)$ denote the number of those $\theta \in \mathrm{Hom}(\mathcal{K}_\lambda/(\mathcal{K}_\lambda \cap \mathcal{H}^2_\lambda), \mathbb{Z}_2)$ such that $\mathrm{rank}\, \theta\circ\omega = r$ where $1 \le r \le \Delta(\lambda)$. Then we obtain an integer vector

$$\mathcal{B}(\lambda) = (b_1(\lambda), ..., b_{\Delta(\lambda)}(\lambda)),$$

called the *bilinear vector*. It is not difficult to see that $\mathcal{B}(\lambda)$ is an invariant of the cohomology ring $H^*(M(\lambda);\mathbb{Z}_2)$.

It should be pointed out that we shall only calculate $b_1(\lambda)$ and $b_2(\lambda)$ in $\mathcal{B}(\lambda)$ because basically this will be sufficient enough to reach our purpose. By $\bar{\mathcal{B}}(\lambda)$ we denote $(b_1(\lambda), b_2(\lambda))$. We also use the convention that $b_2(\lambda) = 0$ if $\Delta(\lambda) = 1$.

6.2. **Calculation of $\Delta(\lambda)$.** First we shall deal with the case in which λ is nontrivial.

Lemma 6.1. *If λ is nontrivial, then*

$$\Delta(\lambda) = \begin{cases} n_\lambda & \text{if } n_\lambda > 0 \text{ and } m_\lambda = 0 \\ n_\lambda - 1 & \text{if } m_\lambda > 0 \\ 1 & \text{if } n_\lambda = 0 \text{ (so } m \text{ is even).} \end{cases}$$

Proof. By Proposition 5.9, each λ is sector-equivalent to the canonical form λ_{C_*} with the coloring sequence $\mathcal{C}_*$ and without changing m_λ and n_λ, so it suffices to consider the λ_{C_*}.

If $n_\lambda > 0$ and $m_\lambda = 0$, we can obtain from (5.4) that λ_{C_*} determines the following three linear relations in $H^1(M(\lambda_{C_*});\mathbb{Z}_2)$

$$c + f = 0 \tag{6.1}$$

$$f + \sum_{i \text{ is odd}} s_i = 0 \tag{6.2}$$

$$s_2 + \cdots + s_{2n_\lambda} + \sum_{2n_\lambda < i \leq m} s_i = 0. \tag{6.3}$$

So we may choose $B_1 = \{f, s_2, s_3, ..., s_{m-1}\}$ as a basis of $H^1(M(\lambda_{C_*}); \mathbb{Z}_2)$. Since $cf = 0$ and $s_i s_j = 0$ with $s_i \cap s_j = \emptyset$ in $H^*(M(\lambda_{C_*}); \mathbb{Z}_2)$, one can easily obtain from (6.1) and (6.3) that $B_2 = \{f, s_2, s_4, ..., s_{2n_\lambda - 2}\} \subset \mathcal{H}^1_{\lambda_{C_*}}$, and $B_2 \subset B_1$. Thus, $\dim \mathcal{H}^1_\lambda = \dim \mathcal{H}^1_{\lambda_{C_*}} \geq n_\lambda$. On the other hand, an easy argument shows that $B_3 = \{s_3^2, ..., s_{2n_\lambda-1}^2, s_{2n_\lambda}^2, ..., s_{m-1}^2, f s_2, s_3 s_4, s_5 s_6, ..., s_{2n_\lambda-1} s_{2n_\lambda}\}$ forms a basis of $H^2(M(\lambda_{C_*}); \mathbb{Z}_2)$. Now observe that the square of each element of $B_1 \setminus B_2$ is in B_3, so $\dim \mathcal{H}^2_\lambda = \dim \mathcal{H}^2_{\lambda_{C_*}} \geq m - 1 - n_\lambda$. Furthermore, $\dim \mathcal{H}^1_\lambda \leq n_\lambda$. Therefore, $\Delta(\lambda) = n_\lambda$.

If $m_\lambda > 0$, then λ_{C_*} determines the following three linear relations

$$\begin{cases} c + f + s_2 + \cdots + s_{2m_\lambda - 2} = 0 \\ f + \sum_{i \text{ is odd}} s_i = 0 \\ s_2 + \cdots + s_{2n_\lambda} + \sum_{2n_\lambda < i \leq m} s_i = 0. \end{cases}$$

In this case, we choose $B_4 = \{s_1, s_2, ..., s_{m-1}\}$ as a basis of $H^1(M(\lambda_{C_*}); \mathbb{Z}_2)$. Then one sees that $B_5 = \{s_2, s_4, ..., s_{2n_\lambda-2}\} \subset \mathcal{H}^1_{\lambda_{C_*}}$. Furthermore, we choose

$$B_6 = \{s_1^2, s_3^2, ..., s_{2n_\lambda-1}^2, s_{2n_\lambda}^2, s_{2n_\lambda+1}^2, ..., s_{m-1}^2, s_2 s_3, s_4 s_5, ..., s_{2n_\lambda-2} s_{2n_\lambda-1}\}$$

as a basis of $H^2(M(\lambda_{C_*}); \mathbb{Z}_2)$. A similar argument as above shows that $\Delta(\lambda) = n_\lambda - 1$.

If $n_\lambda = 0$, in a similar way as above, it is easy to see that we may choose $B_7 = \{f, s_3, ..., s_m\}$ as a basis of $H^1(M(\lambda_{C_*}); \mathbb{Z}_2)$ and $B_8 = \{f\}$ forms a basis of $\mathcal{H}^1_{\lambda_{C_*}}$. Thus, $\Delta(\lambda) = 1$. □

Next we consider the case in which λ is trivial.

Lemma 6.2. *Let λ be trivial. Then*

$$\Delta(\lambda) = \begin{cases} m-1 & \text{if } \lambda \approx \lambda_{C_1} \\ m-2 & \text{if } \lambda \approx \lambda_{C_i}, i = 2,3,4 \\ m-3 & \text{if } \lambda \approx \lambda_{C_i} \text{ with } m > 4,\ i = 5,6,7,8,9,10. \end{cases}$$

In particular, if $m = 3$ then $\Delta(\lambda_{C^3}) = 0$, and if $m = 4$ then $\Delta(\lambda_{C_1^4}) = \Delta(\lambda_{C_2^4}) = 1$.

Proof. The argument is similar to that of Lemma 6.1, and is not quite difficult. Here we only list the three linear relations and the bases of $H^i(M(\lambda); \mathbb{Z}_2)(i = 1, 2)$ and $\mathcal{H}^1_\lambda$, but for the detailed proof, we would like to leave it to readers as an exercise.

λ	m	Three linear relations by determined by J_λ
λ_{C_1}	even	$c+f=0, \sum_{i \text{ is odd}} s_i = 0, \sum_{i \text{ is even}} s_i = 0$
λ_{C_2}	even	$c+f=0, s_2+\sum_{i \text{ is odd}} s_i = 0, \sum_{i \text{ is even}} s_i = 0$
λ_{C_3}	odd	$c+f=0, s_2+\sum_{i \text{ is odd}} s_i = 0, \sum_{i \text{ is even}} s_i = 0$
λ_{C_4}	even	$c+f+s_1=0, \sum_{i \text{ is odd}} s_i = 0, \sum_{i \text{ is even}} s_i = 0$
λ_{C_5}	odd	$c+f+s_1=0, \sum_{i \text{ is odd}} s_i = 0, s_1+\sum_{i \text{ is even}} s_i = 0$
λ_{C_6}	odd	$c+f+s_1+s_2=0, \sum_{i \text{ is odd}} s_i = 0, s_1+\sum_{i \text{ is even}} s_i = 0$
λ_{C_7}	odd	$c+f+s_2=0, \sum_{i \text{ is odd}} s_i = 0, s_1+\sum_{i \text{ is even}} s_i = 0$
λ_{C_8}	even	$c+f+s_1+s_3=0, s_2+\sum_{i \text{ is odd}} s_i = 0, \sum_{i \text{ is even}} s_i = 0$
λ_{C_9}	even	$c+f+s_3=0, s_2+\sum_{i \text{ is odd}} s_i = 0, \sum_{i \text{ is even}} s_i = 0$
$\lambda_{C_{10}}$	even	$c+f+s_2=0, s_2+\sum_{i \text{ is odd}} s_i = 0, \sum_{i \text{ is even}} s_i = 0$

λ	Basis of $H^1(M(\lambda);\mathbb{Z}_2)$	Basis of $\mathcal{H}^1_\lambda$	Basis of $H^2(M(\lambda);\mathbb{Z}_2)$
λ_{C_1}	$\{f, s_3, ..., s_m\}$	$\{f, s_3, ..., s_m\}$	$\{s_3s_4, fs_3, ..., fs_m\}$
λ_{C_2}	$\{f, s_2, ..., s_{m-1}\}$	$\{f, s_2, s_4, ..., s_{m-1}\}$	$\{s_1^2, fs_2, ..., fs_{m-1}\}$
λ_{C_3}	$\{f, s_2, ..., s_{m-1}\}$	$\{f, s_4, ..., s_m\}$	$\{s_1^2, fs_1, ..., fs_{m-2}\}$
λ_{C_4}	$\{f, s_2, ..., s_{m-1}\}$	$\{s_3, ..., s_m\}$	$\{s_1s_2, fs_3, ..., fs_m\}$
λ_{C_5}	$\{f, s_2, ..., s_{m-1}\}$	$\{s_3, ..., s_{m-1}\}$	$\{s_1s_2, fs_2, ..., fs_{m-1}\}$
λ_{C_6}	$\{f, s_2, ..., s_{m-1}\}$	$\{s_3, ..., s_{m-1}\}$	$\{f^2, s_2^2, fs_2, ..., fs_{m-2}\}$
λ_{C_7}	$\{f, s_2, ..., s_{m-1}\}$	$\{s_3, ..., s_{m-1}\}$	$\{f^2, s_2^2, fs_3, ..., fs_{m-1}\}$
λ_{C_8}	$\{f, s_2, ..., s_{m-1}\}$	$\{s_2, s_4, ..., s_{m-1}\}$	$\{f^2, s_3^2, fs_1, fs_4, ..., fs_{m-1}\}$
λ_{C_9}	$\{f, s_2, ..., s_{m-1}\}$	$\{s_2, s_4, ..., s_{m-1}\}$	$\{f^2, s_3^2, fs_1, fs_4, ..., fs_{m-1}\}$
$\lambda_{C_{10}}$	$\{f, s_2, ..., s_{m-1}\}$	$\{s_2, s_4, ..., s_{m-1}\}$	$\{f^2, s_3^2, fs_3, ..., fs_{m-1}\}$

□

Remark 6.1. Although it is not mentioned in this paper, the authors have calculated the first Betti number under $\mathbb{Z}$-coefficients of all small covers over prisms and discovered that the number is always equal to $\Delta(\lambda)$ in the $\mathbb{Z}_2$-cohomology ring. One can check that this is also true for all closed surfaces (i.e., 2-dimensional small covers). It should be reasonable to conjecture that this is true for all small covers.

Proposition 6.3. *Let λ_1 and λ_2 be two colorings in $\Lambda(\mathrm{P}^3(m))$ such that λ_1 is trivial but λ_2 is nontrivial. If $m > 6$, then both $M(\lambda_1)$ and $M(\lambda_1)$ cannot be homeomorphic.*

Proof. Suppose that $M(\lambda_1)$ and $M(\lambda_1)$ are homeomorphic. Then their cohomologies are isomorphic, so $\Delta(\lambda_1) = \Delta(\lambda_2)$. However, by Lemmas 6.1 and 6.2, one has that $\Delta(\lambda_1) \geq m-3$ and $\Delta(\lambda_2) \leq m/2$. Furthermore, if $m > 6$, then $\Delta(\lambda_1) \geq m-3 > m/2 \geq \Delta(\lambda_2)$, so $\Delta(\lambda_1) \neq \Delta(\lambda_2)$, a contradiction. □

Remark 6.2. We see from the proof of Proposition 6.3 that $\Delta(\lambda_1)$ and $\Delta(\lambda_2)$ can coincide only if $m \leq 6$. For $m = 5, 6$, all possible cases that $\Delta(\lambda_1) = \Delta(\lambda_2)$ happens are stated as follows: when $(n_{\lambda_{C_*}}, m_{\lambda_{C_*}}) = (m-3, 0)$, one has that $\Delta(\lambda_{C_*}) = \Delta(\lambda_{C_i})$, $i = 5, 6, 7, 8, 9, 10$. For $m = 3, 4$, we know from [LY] and [M3] that up to homeomorphism, there are only two small covers over $\mathrm{P}^3(3)$: $\mathbb{R}P^3$ and $S^1 \times \mathbb{R}P^2$, and there are only four small covers over $\mathrm{P}^3(4)$: $(S^1)^3$, $S^1 \times K$, a twist $(S^1)^2$-bundle over S^1 and a twist K-bundle over S^1, where K is a Klein bottle. In particular, the cohomological rigidity holds in this case.

6.3. **Calculation of $\bar{\mathcal{B}}(\lambda)$.** Let $\lambda \in \Lambda(\mathrm{P}^3(m))$ with $m > 4$. Choose an ordered basis B' of $H^1(M(\lambda); \mathbb{Z}_2)$ and an ordered basis B'' of $\mathcal{H}^1_\lambda$, let A_0 denote an $(m-1)\times\Delta(\lambda)$ matrix (a_{ij}), where $a_{ij} = [u_i v_j] \in \mathcal{K}_\lambda/(\mathcal{K}_\lambda \cap \mathcal{H}^2_\lambda)$, u_i is the i-th element in B' and v_j the j-th element in B'', so each element in B' corresponds to a row and each element in B'' a column. It follows that for any $\theta \in \mathrm{Hom}(\mathcal{K}_\lambda/(\mathcal{K}_\lambda \cap \mathcal{H}^2_\lambda), \mathbb{Z}_2)$, $\theta(A_0) = (\theta(a_{ij}))$ is a representation matrix of $\theta \circ \omega$.

First let us look at the case in which λ is nontrivial.

Lemma 6.4. *Let λ be nontrivial. Then*

$$\bar{\mathcal{B}}(\lambda) = \begin{cases} (0,0) & \text{if } (n_\lambda, m_\lambda) = (0,0) \\ (1,0) & \text{if } (n_\lambda, m_\lambda) = (1,0) \text{ or } (2,2) \\ (1,3) & \text{if } (n_\lambda, m_\lambda) = (2,0) \\ (0, n_\lambda) & \text{if } n_\lambda > 2 \text{ and } m_\lambda = 0 \\ (n_\lambda - m_\lambda, \binom{m_\lambda - 1}{1} + \binom{m_\lambda - 1}{2} + \binom{n_\lambda - m_\lambda}{2}) & \text{if } n_\lambda > 2 \text{ and } m_\lambda > 0 \end{cases}$$

Proof. By Proposition 5.9, one may assume that $\lambda = \lambda_{C_*}$. Then our argument proceeds as follows.

(1) If $n_\lambda > 0$ and $m_\lambda = 0$, then Lemma 6.1 we may take $B' = B_1$ and $B'' = B_2$. Thus one has that A_0 is equal to

$$\begin{bmatrix}
0 & [fs_2] & [fs_4] & [fs_6] & \cdots & [fs_{2n_\lambda-6}] & [fs_{2n_\lambda-4}] & [fs_{2n_\lambda-2}] \\
[s_2 f] & 0 & 0 & 0 & \cdots & \cdots & \cdots & 0 \\
[s_3 f] & [s_3 s_2] & [s_3 s_4] & 0 & \cdots & \cdots & \cdots & 0 \\
[s_4 f] & 0 & 0 & 0 & \cdots & \cdots & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
[s_{2n_\lambda-3} f] & 0 & \cdots & \cdots & \cdots & 0 & [s_{2n_\lambda-3} s_{2n_\lambda-4}] & [s_{2n_\lambda-3} s_{2n_\lambda-2}] \\
[s_{2n_\lambda-2} f] & 0 & 0 & 0 & \cdots & \cdots & \cdots & 0 \\
[s_{2n_\lambda-1} f] & 0 & 0 & 0 & \cdots & \cdots & 0 & [s_{2n_\lambda-1} s_{2n_\lambda-2}] \\
[s_{2n_\lambda} f] & 0 & 0 & 0 & \cdots & \cdots & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
[s_{m-1} f] & 0 & 0 & 0 & \cdots & \cdots & \cdots & 0
\end{bmatrix}$$

By direct calculations one knows from (6.2) and (6.3) that $s_{2n_\lambda} s_{2n_\lambda+1} + s^2_{2n_\lambda+1} + \cdots + s^2_m = 0$ so $[s_{2n_\lambda} s_{2n_\lambda+1}] = 0$ and

$$\begin{cases} s_{2i} s_{2i+1} = s_{2i+1} s_{2i+2} & \text{when } 1 \le i \le n_\lambda - 1 \\ f s_i = s_i^2 \text{ so } [f s_i] = 0 & \text{when either } i \text{ is odd or } i > 2n_\lambda \text{ is even} \\ f s_i = s_{i-1} s_i + s_i s_{i+1} & \text{when } 2 \le i \le 2n_\lambda \text{ is even.} \end{cases}$$

Set $x_1 = [fs_2]$ and $x_i = [s_{2i-1} s_{2i}]$ for $2 \le i \le n_\lambda$. Then $[fs_{2i}] = x_i + x_{i+1}$ for $2 \le i \le n_\lambda - 1$ and $[fs_{2n_\lambda}] = x_{n_\lambda}$. Thus, we see that $\{x_1, ..., x_{n_\lambda}\}$ forms a basis of $\mathcal{K}_\lambda/(\mathcal{K}_\lambda \cap \mathcal{H}^2_\lambda)$, and the corresponding rows of $f, s_2, ..., s_{2n_\lambda}$ in A_0 are nonzero. Now we may reduce A_0 to A by deleting those zero rows of A_0, so that for each $\theta \in \mathrm{Hom}(\mathcal{K}_\lambda/(\mathcal{K}_\lambda \cap \mathcal{H}^2_\lambda), \mathbb{Z}_2)$, $\mathrm{rank}(\theta(A_0)) = \mathrm{rank}(\theta(A))$ still holds. Write A as follows:

$$\begin{bmatrix}
0 & x_1 & x_2 + x_3 & x_3 + x_4 & \cdots & x_{n_\lambda-3} + x_{n_\lambda-2} & x_{n_\lambda-2} + x_{n_\lambda-1} & x_{n_\lambda-1} + x_{n_\lambda} \\
x_1 & 0 & 0 & 0 & \cdots & \cdots & \cdots & 0 \\
0 & x_2 & x_2 & 0 & \cdots & \cdots & \cdots & 0 \\
x_2 + x_3 & 0 & 0 & 0 & \cdots & \cdots & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & \cdots & \cdots & 0 & x_{n_\lambda-1} & x_{n_\lambda-1} \\
x_{n_\lambda-1} + x_{n_\lambda} & 0 & 0 & 0 & \cdots & \cdots & \cdots & 0 \\
0 & 0 & 0 & 0 & \cdots & \cdots & 0 & x_{n_\lambda} \\
x_{n_\lambda} & 0 & 0 & 0 & \cdots & \cdots & \cdots & 0
\end{bmatrix}$$

Let $\{\theta_i \big| i=1,...,n_\lambda\}$ be the dual basis of $\{x_1,...,x_{n_\lambda}\}$ in $\mathrm{Hom}(\mathcal{K}_\lambda/(\mathcal{K}_\lambda\cap\mathcal{H}_\lambda^2),\mathbb{Z}_2)$. Take any $\theta\in\mathrm{Hom}(\mathcal{K}_\lambda/(\mathcal{K}_\lambda\cap\mathcal{H}_\lambda^2),\mathbb{Z}_2)$, one may write $\theta=\sum_{i\in S}\theta_i$ where $S\subset\{1,...,n_\lambda\}$. Obviously, if $n_\lambda=1$ then $b_1(\lambda)=1$ and $b_2(\lambda)=0$. If $n_\lambda\geq 2$, it is easy to see that $b_1(\lambda)=0$ since $\mathrm{rank}\theta(A)$ cannot be 1 whenever S is empty or non-empty. If $n_\lambda=2$, then $\mathrm{rank}\theta(A)=2$ only when $S=\{1\},\{2\},\{1,2\}$, so $b_2(\lambda)=3$. If $n_\lambda>2$, by direct calculations, one has that only when $S=\{i\}(i\neq 2)$ or $\{1,2\}$, $\mathrm{rank}\theta(A)=2$, so $b_2(\lambda)=n_\lambda$.

(2) If $n_\lambda>0$, then Lemma 6.1 we may take $B'=B_4$ and $B''=B_5$. Moreover, we see that in A_0, only corresponding rows of $s_1,s_3,...,s_{2n_\lambda-1}$ are nonzero, so we may delete the other rows from A_0 to obtain A so that for each $\theta\in\mathrm{Hom}(\mathcal{K}_\lambda/(\mathcal{K}_\lambda\cap\mathcal{H}_\lambda^2),\mathbb{Z}_2)$, $\mathrm{rank}(\theta(A_0))=\mathrm{rank}(\theta(A))$. Now we can write down A after simple calculations:

$$A=\begin{bmatrix}[s_1s_2] & 0 & \cdots\cdots & & 0\\ [s_2s_3] & [s_2s_3] & 0 & \cdots\cdots & 0\\ 0 & [s_4s_5] & [s_4s_5] & 0 & \cdots\cdots\\ \cdots\cdots & \cdots & \cdots & \cdots & \cdots\cdots\\ \cdots\cdots & 0 & & [s_{2n_\lambda-4}s_{2n_\lambda-3}] & [s_{2n_\lambda-4}s_{2n_\lambda-3}]\\ \cdots\cdots & & & 0 & [s_{2n_\lambda-2}s_{2n_\lambda-1}]\end{bmatrix}$$

Set $x_i=[s_{2i}s_{2i+1}]\in\mathcal{K}_\lambda/(\mathcal{K}_\lambda\cap\mathcal{H}_\lambda^2)$, for $i=1,...,n_\lambda-1$. A direct calculation shows that $[s_1s_2]=x_1+x_2+\cdots+x_{m_\lambda-1}$. So we see that $\{x_i\big|i=1,...,n_\lambda-1\}$ forms a basis of $\mathcal{K}_\lambda/(\mathcal{K}_\lambda\cap\mathcal{H}_\lambda^2)$. Let $\{\theta_i\big|i=1,...,n_\lambda-1\}$ be its dual basis in $\mathrm{Hom}(\mathcal{K}_\lambda/(\mathcal{K}_\lambda\cap\mathcal{H}_\lambda^2),\mathbb{Z}_2)$. Then one may write $\theta=\sum_{i\in S}\theta_i$ where $S\subset\{1,...,n_\lambda-1\}$. Now $i\in S$ implies that the $(i+1)$-th row of $\theta(A)$ is nonzero. In order that $\mathrm{rank}(\theta(A))=1$, one must have $\sharp(S)=1$ since S cannot be empty. If $n_\lambda=2$ then $m_\lambda=2$, $\mathcal{H}_\lambda^1$ is 1-dimensional and $\mathcal{K}_\lambda/(\mathcal{K}_\lambda\cap\mathcal{H}_\lambda^2)$ has only a nonzero element, so $b_1(\lambda)=1$ and $b_2(\lambda)=0$. If $n_\lambda>2$, $\mathrm{rank}(\theta_i(A))=1$ if and only if $\theta_i([s_1s_2])=0$, which is equivalent to that $\theta_i(x_1+x_2+...+x_{m_\lambda-1})=0\Leftrightarrow i>m_\lambda-1$. Therefore, $b_1(\lambda)=n_\lambda-m_\lambda$. In this case, an easy argument shows that $b_2(\lambda)=\binom{m_\lambda-1}{1}+\binom{m_\lambda-1}{2}+\binom{n_\lambda-m_\lambda}{2}$.

(3) If $n_\lambda=0$, then Lemma 6.1 we may take $B'=B_4$ and $B''=B_5$, so

$$A_0=(0,[s_3f],...,[s_mf]).$$

However, a direct calculation shows that for each i, $s_if=s_i^2$, so $[s_if]=0$ in $\mathcal{K}_\lambda/(\mathcal{K}_\lambda\cap\mathcal{H}_\lambda^2)$. Thus, $\dim\mathcal{K}_\lambda/(\mathcal{K}_\lambda\cap\mathcal{H}_\lambda^2)=0$, and so $\bar{\mathcal{B}}(\lambda)=(0,0)$. □

Theorem 6.5. *Let λ_1,λ_2 be two nontrivial colorings on $\mathrm{P}^3(m)$ with $m>4$. Then $M(\lambda_1)$ and $M(\lambda_2)$ are homeomorphic if and only if their cohomologies $H^*(M(\lambda_1);\mathbb{Z}_2)$ and $H^*(M(\lambda_2);\mathbb{Z}_2)$ are isomorphic as rings.*

Proof. It suffices to show that if $H^*(M(\lambda_1);\mathbb{Z}_2)$ and $H^*(M(\lambda_2);\mathbb{Z}_2)$ are isomorphic, then $M(\lambda_1)$ and $M(\lambda_2)$ are homeomorphic. Now suppose that $H^*(M(\lambda_1);\mathbb{Z}_2)\cong H^*(M(\lambda_2);\mathbb{Z}_2)$. Then one has that $\bar{\mathcal{B}}(\lambda_1)=\bar{\mathcal{B}}(\lambda_2)$. We claim that $(m_{\lambda_1},n_{\lambda_1})=(m_{\lambda_2},n_{\lambda_2})$. If not, then by Lemma 6.4, the possible case in which this happens is $\bar{\mathcal{B}}(\lambda_1)=\bar{\mathcal{B}}(\lambda_2)=(1,0)$. Without loss of generality, assume that $(m_{\lambda_1},n_{\lambda_1})=(1,0)$ and $(m_{\lambda_2},n_{\lambda_2})=(2,2)$. Then by Lemma 6.1, one has $\Delta(\lambda_1)=\Delta(\lambda_2)=1$, so $\mathcal{H}_{\lambda_1}^1$ and $\mathcal{H}_{\lambda_2}^1$ contains only a nonzero element. Let $z_0^{(i)}$ be the unique nonzero element of

$\mathcal{H}^1_{\lambda_i}, i=1,2$. For each i, define a linear map $\Phi_i : H^1(M(\lambda_i);\mathbb{Z}_2) \longrightarrow H^2(M(\lambda_i);\mathbb{Z}_2)$ by $x \longmapsto z_0^{(i)}x$.

When $i=1$, by Lemma 6.1 one may choose $B_1 = \{f, s_2, s_3, ..., s_{m-1}\}$ as a basis of $H^1(M(\lambda_1);\mathbb{Z}_2)$ and $B_2 = \{f\}$ as a basis of $\mathcal{H}^1_{\lambda_1}$, so $z_0^{(1)} = f$. By direct calculations, one has that for $3 \le j \le m-1$, $fs_j = s_j^2$. Since $fs_2, s_3^2, ..., s_{m-1}^2$ are linearly independent, one knows that Φ_1 has rank $m-2$.

When $i=2$, by Lemma 6.1 one may choose $B_4 = \{s_1, s_2, ..., s_{m-1}\}$ as a basis of $H^1(M(\lambda_2);\mathbb{Z}_2)$ and $B_5 = \{s_2\}$ as a basis of $\mathcal{H}^1_{\lambda_2}$, so $z_0^{(1)} = s_2$. Since $s_2^2 = s_2s_j = 0, j \ge 4$, one sees that Φ_2 has rank at most 2.

Now since $m > 4$, one has that $\mathrm{rank}\Phi_1 = m-2 > 2 \ge \mathrm{rank}\Phi_2$, but this is impossible. Thus, one must have $(m_{\lambda_1}, n_{\lambda_1}) = (m_{\lambda_2}, n_{\lambda_2})$. Moreover, the theorem follows from Corollary 5.10. □

Corollary 6.6. *Let λ_1, λ_2 be two nontrivial colorings on $\mathrm{P}^3(m)$ with $m > 4$. Then $M(\lambda_1)$ and $M(\lambda_2)$ are homeomorphic if and only if $(m_{\lambda_1}, n_{\lambda_1}) = (m_{\lambda_2}, n_{\lambda_2})$.*

Furthermore, by Corollary 5.11 one has

Corollary 6.7. *The number of homeomorphism classes of small covers over $\mathrm{P}^3(m)$ $(m > 4)$ with nontrivial colorings is exactly*

$$N_{nt}(m) = \begin{cases} \sum_{0\le k\le \frac{m}{2}}([\frac{k}{2}]+1) & \text{if } m \text{ is even} \\ \sum_{1\le k\le \frac{m}{2}}([\frac{k}{2}]+1) & \text{if } m \text{ is odd.} \end{cases}$$

Next let us look at the case in which λ is trivial. By Lemma 6.2 we divide our argument into two cases: (I) $\Delta(\lambda)$ is odd; (II) $\Delta(\lambda)$ is even.

Case (I): $\Delta(\lambda)$ is odd.

Lemma 6.8. *Let λ be trivial such that $\Delta(\lambda)$ is odd. Then*

$$\bar{\mathcal{B}}(\lambda) = \begin{cases} (0, 2^{m-2}-1) & \text{if } \lambda \approx \lambda_{C_1} \\ (0, 2^{m-3}-1) & \text{if } \lambda \approx \lambda_{C_3} \\ (2^{m-4}-1, 0) & \text{if } \lambda \approx \lambda_{C_8} \\ (2^{m-3}-1, 0) & \text{if } \lambda \approx \lambda_{C_9} \\ (2^{m-4}-1, 0) & \text{if } \lambda \approx \lambda_{C_{10}} \end{cases}$$

Proof. If $\lambda \approx \lambda_{C_1}$, using Lemma 6.2 and by direct calculations, one has that $s_1s_2 = s_2s_3 = \cdots = s_{m-1}s_m = s_ms_1$, so A_0 may be written as follows:

$$\begin{bmatrix} 0 & x_2 & x_3 & x_4 & \cdots & x_{m-2} & x_{m-1} \\ x_2 & 0 & x_1 & 0 & \cdots & 0 & 0 \\ x_3 & x_1 & 0 & x_1 & \cdots & 0 & 0 \\ x_4 & 0 & x_1 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ x_{m-2} & 0 & 0 & 0 & \cdots & 0 & x_1 \\ x_{m-1} & 0 & 0 & 0 & \cdots & x_1 & 0 \end{bmatrix}$$

where $x_1 = [s_3s_4]$ and $x_i = [fs_{i+1}], i = 2, ..., m-1$. We see easily that $\{x_1, ..., x_{m-1}\}$ forms a basis of $\mathcal{K}_\lambda/(\mathcal{K}_\lambda \cap \mathcal{H}^2_\lambda)$ so $\dim \mathcal{K}_\lambda/(\mathcal{K}_\lambda \cap \mathcal{H}^2_\lambda) = m-1$. Then one may conclude that $\bar{\mathcal{B}}(\lambda) = (0, 2^{m-2}-1)$. Also, it is easy to see that in this case $b_{\Delta(\lambda)}$ is nonzero.

In a similar way as above, if $\lambda \approx \lambda_{C_3}$, one has that $[s_2s_3] = [s_3s_4] = [s_4s_5] = \cdots = [s_{m-1}s_m] = 0$, so A_0 may be written as follows:

$$\begin{bmatrix} 0 & x_3 & x_4 & x_5 & \cdots & x_{m-3} & \sum_{j \text{ is odd}} x_j & x_1 + \sum_{j \text{ is even}} x_j \\ x_1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ x_2 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ x_3 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ x_{m-3} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \sum_{j \text{ is odd}} x_j & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}$$

where $x_i = [fs_{i+1}], i = 1, ..., m-3$. And $\{x_1, ..., x_{m-3}\}$ forms a basis of $\mathcal{K}_\lambda/(\mathcal{K}_\lambda \cap \mathcal{H}^2_\lambda)$ so $\dim \mathcal{K}_\lambda/(\mathcal{K}_\lambda \cap \mathcal{H}^2_\lambda) = m-3$. A direct observation shows that $\bar{\mathcal{B}}(\lambda) = (0, 2^{m-3}-1)$.

If $\lambda \approx \lambda_{C_8}$ or λ_{C_9}, then $[s_1s_2] = [s_2s_3] = \cdots = [s_{m-1}s_m] = [s_ms_1] = 0$, so A_0 can be reduced to a $1 \times (m-3)$ matrix

$$([fs_2], [fs_4], ..., [fs_{m-1}]).$$

Also, we easily see that $\{s_3^2, fs_2, fs_4, ..., fs_{m-1}\}$ can be used as a basis of $\mathcal{K}_\lambda$ and $\{f^2, s_3^2\}$ forms a basis of $\mathcal{H}^2_\lambda$ (note that $\dim \mathcal{H}^2_\lambda = m - 1 - \Delta(\lambda)$=2). However, when $\lambda \approx \lambda_{C_8}$, by direct calculations one has that $f^2 = fs_2 + \sum_{j>4 \text{ is odd}} fs_j$, so $f^2 \in \mathcal{K}_\lambda$ and $\mathcal{H}^2_\lambda \subset \mathcal{K}_\lambda$. Thus, $\dim \mathcal{H}^2_\lambda \cap \mathcal{K}_\lambda = 2$, $\dim \mathcal{K}_\lambda/(\mathcal{K}_\lambda \cap \mathcal{H}^2_\lambda) = m-4$ and $\{[fs_4], ..., [fs_{m-1}]\}$ forms a basis of $\mathcal{K}_\lambda/(\mathcal{K}_\lambda \cap \mathcal{H}^2_\lambda)$. Moreover, one has that $\bar{\mathcal{B}}(\lambda_{C_8}) = (2^{m-4}-1, 0)$. When $\lambda \approx \lambda_{C_9}$, it is not difficult to check that $\dim \mathcal{H}^2_\lambda \cap \mathcal{K}_\lambda = 1$ and $[fs_2], [fs_4], ..., [fs_{m-1}]$ are linearly independent, so $\dim \mathcal{K}_\lambda/(\mathcal{K}_\lambda \cap \mathcal{H}^2_\lambda) = m-3$. Thus, $\bar{\mathcal{B}}(\lambda_{C_9}) = (2^{m-3}-1, 0)$.

If $\lambda \approx \lambda_{C_{10}}$, then $[s_1s_2] = [s_2s_3] = \cdots = [s_{m-1}s_m] = [s_ms_1] = [s_3^2] = 0$ and $fs_2 = f^2$, so A_0 can be reduced to a $1 \times (m-3)$ matrix $(0, [fs_4], ..., [fs_{m-1}])$. It is easy to see that $\dim \mathcal{H}^2_\lambda \cap \mathcal{K}_\lambda = 1$ and $\dim \mathcal{K}_\lambda/(\mathcal{K}_\lambda \cap \mathcal{H}^2_\lambda) = m-4$, so $\bar{\mathcal{B}}(\lambda_{C_{10}}) = (2^{m-4}-1, 0)$. □

Case (II): $\Delta(\lambda)$ is even.

Lemma 6.9. *Let λ be trivial such that $\Delta(\lambda)$ is even. Then*

$$\bar{\mathcal{B}}(\lambda) = \begin{cases} (1, 2^{m-2}-2) & \text{if } \lambda \approx \lambda_{C_2} \\ (2^{m-2}-1, 0) & \text{if } \lambda \approx \lambda_{C_4} \\ (2^{m-3}-1, 0) & \text{if } \lambda \approx \lambda_{C_5} \\ (2^{m-4}-1, 0) & \text{if } \lambda \approx \lambda_{C_6} \\ (2^{m-3}-1, 0) & \text{if } \lambda \approx \lambda_{C_7} \end{cases}$$

Proof. If $\lambda \approx \lambda_{C_2}$, then one can obtain by Lemma 6.2 that $[s_1s_2] = [s_2s_3] = \cdots = [s_{m-1}s_m] = [s_ms_1] = [s_1^2] = 0$ and so A_0 can be reduced to the following matrix

$$\begin{bmatrix} 0 & [fs_2] & [fs_4] & \cdots & [fs_{m-1}] \\ [fs_2] & 0 & 0 & \cdots & 0 \\ [fs_3] & 0 & 0 & \cdots & 0 \\ [fs_4] & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ [fs_{m-1}] & 0 & 0 & \cdots & 0 \end{bmatrix}$$

One may easily show that $\{[fs_2], ..., [fs_{m-1}]\}$ is a basis of $\mathcal{K}_\lambda/(\mathcal{K}_\lambda \cap \mathcal{H}^2_\lambda)$. Then a direct observation can obtain that $\bar{\mathcal{B}}(\lambda_{C_2}) = (1, 2^{m-2} - 2)$.

If $\lambda \approx \lambda_{C_4}$, then one has that $[s_1 s_2] = [s_2 s_3] = \cdots = [s_{m-1} s_m]$, so A_0 can be reduced to the following matrix

$$\begin{bmatrix} [fs_3] & [fs_4] & [fs_5] & \cdots & [fs_{m-3}] & [fs_{m-2}] & [fs_{m-1}] & [fs_m] \\ [s_1s_2] & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & [s_1s_2] & 0 & \cdots & 0 & 0 & 0 & 0 \\ [s_1s_2] & 0 & [s_1s_2] & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & [s_1s_2] & 0 & [s_1s_2] & 0 \\ 0 & 0 & 0 & \cdots & 0 & [s_1s_2] & 0 & [s_1s_2] \end{bmatrix}$$

and $\{[s_1 s_2], [fs_3], ..., [fs_m]\}$ is a basis of $\mathcal{K}_\lambda/(\mathcal{K}_\lambda \cap \mathcal{H}^2_\lambda)$ so $\dim \mathcal{K}_\lambda/(\mathcal{K}_\lambda \cap \mathcal{H}^2_\lambda) = m-1$. Furthermore, one knows that $\bar{\mathcal{B}}(\lambda_{C_4}) = (2^{m-2} - 1, 0)$. Note that in this case $b_{\Delta(\lambda)}$ is nonzero.

If $\lambda \approx \lambda_{C_5}$, then one has that $[s_1^2] = [s_1 s_2] = [s_2 s_3] = \cdots = [s_{m-1} s_m] = 0$, so A_0 can be reduced to a $1 \times (m-3)$ matrix $([fs_3], [fs_4], ..., [fs_{m-1}])$, and $\{[fs_3], [fs_4], ..., [fs_{m-1}]\}$ is a basis of $\mathcal{K}_\lambda/(\mathcal{K}_\lambda \cap \mathcal{H}^2_\lambda)$ so $\dim \mathcal{K}_\lambda/(\mathcal{K}_\lambda \cap \mathcal{H}^2_\lambda) = m-3$. Note that in this case $\dim \mathcal{H}^2_\lambda \cap \mathcal{K}_\lambda = 1$. Thus, $\bar{\mathcal{B}}(\lambda_{C_5}) = (2^{m-3} - 1, 0)$, but $b_{\Delta(\lambda)} = 0$.

If $\lambda \approx \lambda_{C_6}$ or λ_{C_7}, similarly to the case $\lambda \approx \lambda_{C_5}$, then one has that $[s_1^2] = [s_1 s_2] = [s_2 s_3] = \cdots = [s_{m-1} s_m] = 0$, so A_0 can be reduced to a $1 \times (m-3)$ matrix

$$([fs_3], [fs_4], ..., [fs_{m-1}]).$$

As in the proof of cases $\lambda \approx \lambda_{C_8}$ or λ_{C_9}, we see that $\{s_2^2, fs_3, fs_4, ..., fs_{m-1}\}$ can be used as a basis of $\mathcal{K}_\lambda$ and $\{f^2, s_2^2\}$ forms a basis of $\mathcal{H}^2_\lambda$. However, when $\lambda \approx \lambda_{C_6}$, it is easy to check that $\mathcal{H}^2_\lambda \subset \mathcal{K}_\lambda$, so $\dim \mathcal{H}^2_\lambda \cap \mathcal{K}_\lambda = 2$ and $\dim \mathcal{K}_\lambda/(\mathcal{K}_\lambda \cap \mathcal{H}^2_\lambda) = m-4$. Moreover, $\bar{\mathcal{B}}(\lambda_{C_6}) = (2^{m-4} - 1, 0)$ and $b_{\Delta(\lambda)} = 0$. When $\lambda \approx \lambda_{C_7}$, one may check that $\dim \mathcal{K}_\lambda \cap \mathcal{H}^2_\lambda = 1$ and then $\dim \mathcal{K}_\lambda/(\mathcal{K}_\lambda \cap \mathcal{H}^2_\lambda) = m-3$. Thus $\bar{\mathcal{B}}(\lambda_{C_7}) = (2^{m-3} - 1, 0)$ and $b_{\Delta(\lambda)} = 0$. □

Remark 6.3. We see that for λ_{C_5} and λ_{C_7}, $\Delta(\lambda_{C_5}) = \Delta(\lambda_{C_7})$ and $\bar{\mathcal{B}}(\lambda_{C_5}) = \bar{\mathcal{B}}(\lambda_{C_7})$. However, we can still distinguish them by using the first Stiefel-Whitney class. Let $w_1(\lambda) \in H^1(M(\lambda); \mathbb{Z}_2)$ denote the first Stiefel-Whitney class. It is well-known that $w_1(\lambda) = 0$ if and only if $M(\lambda)$ is orientable. Then, by Corollary 5.6 one knows that if $\lambda \approx \lambda_{C_5}$, then $w_1(\lambda_{C_5}) = 0$; but if $\lambda \approx \lambda_{C_7}$, $w_1(\lambda_{C_7}) \neq 0$. This also happens for λ_{C_8} and $\lambda_{C_{10}}$. But we can use the number $\dim \mathcal{K}_\lambda \cap \mathcal{H}^2_\lambda$ to distinguish them. Actually, by Lemma 6.8, if $\lambda \approx \lambda_{C_8}$, $\dim \mathcal{K}_\lambda \cap \mathcal{H}^2_\lambda = 2$; but if $\lambda \approx \lambda_{C_{10}}$, $\dim \mathcal{K}_\lambda \cap \mathcal{H}^2_\lambda = 1$.

Theorem 6.10. *Let λ_1, λ_2 be two trivial colorings on $\mathrm{P}^3(m)$ with $m > 4$. Then $M(\lambda_1)$ and $M(\lambda_2)$ are homeomorphic if and only if their cohomologies $H^*(M(\lambda_1); \mathbb{Z}_2)$ and $H^*(M(\lambda_2); \mathbb{Z}_2)$ are isomorphic as rings.*

Proof. This follows immediately from Lemmas 6.2, 6.8-6.9 and Remark 6.3. □

As a consequence of Collorary 5.5 and Theorem 6.10, one has

Corollary 6.11. *The number of homeomorphism classes of small covers over* $\mathrm{P}^3(m)(m > 4)$ *with trivial colorings is exactly*

$$N_t(m) = \begin{cases} 4 & \textit{if } m \textit{ is odd} \\ 6 & \textit{if } m \textit{ is even} \end{cases}$$

7. Proofs of Theorems 1.1 and 1.2

Now let us finish the proofs of Theorems 1.1 and 1.2.

Proof of Theorem 1.1. It suffices to show that if their cohomologies $H^*(M(\lambda_1); \mathbb{Z}_2)$ and $H^*(M(\lambda_2); \mathbb{Z}_2)$ are isomorphic as rings, then $M(\lambda_1)$ and $M(\lambda_2)$ are homeomorphic. By Propositions 6.3, 6.5 and 6.10, this is true when $m > 6$. It remains to consider the case $m \leq 6$. As stated in Remark 6.2, the cohomological rigidity holds when $m \leq 4$ (see also [LY] and [M3]). Next, we only need put our attention on the case $5 \leq m \leq 6$. By Lemmas 6.1, 6.2, 6.4, 6.8, 6.9 and Remark 6.3, we may list all possible λ with mentioned invariants in the case $5 \leq m \leq 6$ whichever λ is trivial or nontrivial.

(A) Case $m = 5$:

λ	Trivialization	$\Delta(\lambda)$	$\bar{\mathcal{B}}(\lambda)$	(n_λ, m_λ)	$w_1(\lambda)$
λ_{C_*}	nontrivial	1	$(1,0)$	$(1,0)$	
λ_{C_*}	nontrivial	2	$(1,3)$	$(2,0)$	
λ_{C_*}	nontrivial	1	$(1,0)$	$(2,2)$	
λ_{C_3}	trivial	3	$(0,3)$		
λ_{C_5}	trivial	2	$(3,0)$		0
λ_{C_6}	trivial	2	$(1,0)$		
λ_{C_7}	trivial	2	$(3,0)$		nonzero

(B) Case $m = 6$:

λ	Trivialization	$\Delta(\lambda)$	$\bar{\mathcal{B}}(\lambda)$	(n_λ, m_λ)	$\dim \mathcal{K}_\lambda \cap \mathcal{H}^2_\lambda$
λ_{C_*}	nontrivial	1	$(1,0)$	$(0,0)$	
λ_{C_*}	nontrivial	1	$(1,0)$	$(1,0)$	
λ_{C_*}	nontrivial	2	$(1,3)$	$(2,0)$	
λ_{C_*}	nontrivial	3	$(0,3)$	$(3,0)$	
λ_{C_*}	nontrivial	1	$(1,0)$	$(2,2)$	
λ_{C_*}	nontrivial	2	$(1,1)$	$(3,2)$	
λ_{C_1}	trivial	5	$(0,15)$		
λ_{C_2}	trivial	4	$(1,14)$		
λ_{C_4}	trivial	4	$(15,0)$		
λ_{C_8}	trivial	3	$(3,0)$		2
λ_{C_9}	trivial	3	$(7,0)$		
$\lambda_{C_{10}}$	trivial	3	$(3,0)$		1

We clearly see from two tables above that by using invariants $\Delta(\lambda)$, $\bar{\mathcal{B}}(\lambda)$, (n_λ, m_λ), $w_1(\lambda)$ and $\dim \mathcal{K}_\lambda \cap \mathcal{H}^2_\lambda$, we can distinguish all $M(\lambda)$ up to homeomorphism when $m = 5, 6$. This completes the proof. $\square$

Furthermore, Theorem 1.2 follows immediately from Theorem 1.1, Corollaries 6.7, 6.11 and Remark 6.2.

Finally, let us return to the invariants $\Delta(\lambda)$ and $\mathcal{B}(\lambda)$ again. We see that generally these invariants can always be defined for any small cover over a simple convex polytop P^n. We would like to pose the following problems:

- Under what condition can $\Delta(\lambda)$ and $\mathcal{B}(\lambda)$ become the combinatorial invariants?
- If $\Delta(\lambda)$ and $\mathcal{B}(\lambda)$ are the combinatorial invariants, then how can one calculate them in terms of polytopes P^n?

References

[BP] V.M. Buchstaber and T.E. Panov, *Torus actions and their applications in topology and combinatorics*, University Lecture Series, vol. **24**, Amer. Math. Soc., Providence, RI, 2002.

[CCL] M. Z. Cai, X. Chen, and Z. Lü, *Small covers over prisms*, Topology Appl. **154** (2007) 2228-2234.

[C] S. Choi, *The number of small covers over cubes*, Algebr. Geom. Topol. **8** (2008), 2391-2399.

[D] M. W. Davis, *Groups generated by reflections and aspherical manifolds not covered by Euclidean space*, Annals of Mathematics, **117** (1983), 293-324.

[DJ] M. W. Davis and T. Januszkiewicz, *Convex polytopes, Coxeter orbifolds and torus Actions*, Duke Mahtematicial Journal, **62** (1991), 417-451.

[GS] A. Garrison, R. Scott, *Small covers of the dodecahedron and the 120-cell*, Proc. Amer. Math. Soc.**131** (2002) 963-971.

[I] I. V. Izmestiev, *Three-dimensional manifolds defined by a coloring of the faces of a simple polytope*, Math. Notes **69** (2001), no. 3-4, 340–346.

[KM] Y. Kamishima and M. Masuda, *Cohomological rigidity of real Bott manifolds*, arXiv:0807.4263.

[LM] Z. Lü and M. Masuda, *Equivariant classification of* 2*-torus manifolds*, arXiv:0802.2313.

[LY] Z. Lü and L. Yu, *Topological types of 3-dimensianl small covers* (accepted), to appear in Forum Math., arXiv:0710.4496.

[M1] M. Masuda, *Equivariant cohomology distinguishes toric manifolds*, Adv. Math. **218** (2008), 2005-2012.

[M2] M. Masuda, *Cohomological non-rigidity of generalized real Bott manifolds of height 2*, preprint, arXiv:0809.2215.

[M3] M. Masuda, *Classification of real Bott manifolds*, preprint, arXiv:0809.2178.

[NN] H. Nakayama and Y. Nishimura, *The orientability of small covers and coloring simple polytopes*, Osaka J. Math. **42** (2005), 243-256.

[Z] G.M. Ziegler, *Lectures on Polytopes*, Graduate Texts in Math., Springer-Verlag, Berlin, 1994.

School of Mathematical Sciences, Fudan University, Shanghai, China
E-mail address: xiangyu.cao08@gmail.com

Institute of Mathematics, School of Mathematical Sciences, Fudan University, Shanghai, 200433, P.R.China.
E-mail address: zlu@fudan.edu.cn